

Analysis of resonance production using relativistic Gamow vectors

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Abstract

The calculation of an amplitude involving resonance production is presented. This calculation employs for the resonance state a relativistic Gamow vector. It is used for investigating the question of compatibility of the relativistic Gamow vectors kinematics, defined by real 4-velocities and complex mass, with the stable particle kinematics; or in other words, the integration of the Gamow vectors with the conventional Dirac bra-ket formalism. The calculation demonstrates a consistent framework comprising stable and Gamow vectors.

1 Introduction

The description of unstable particles with state vectors stems from the point of view that unstable particles are not less fundamental than the stable ones. In fact, unstable particles are listed along the stable ones in the Particle Data Table [1] and attributed values for the mass, the spin and the width (or lifetime). Hence a zero value for the width (or an infinite lifetime) is what distinguishes a stable from an unstable particle. The relativistic Gamow vectors provide state vectors for unstable particles through a precise formulation of complex mass representations of the Poincaré group with self-adjoint generators [2, 3]. These representations are “minimally complex”, in the sense that, while the mass is complex, the 4-velocity is real. Furthermore, they

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have the remarkable feature of being representations for the causal Poincaré semigroup into the forward light cone, hence an exponential decay law defined only in the forward light cone.

The relativistic Gamow vectors have all the properties that are expected for state vectors describing unstable particles. We explore in what follows the kinematical implications of the characteristic properties of the relativistic Gamow vectors, namely those of real 4-velocities and complex mass. In other words, the question of integrating these properties with the standard kinematics of stable particles into a consistent framework is addressed. The investigation is carried here by analyzing an example of resonance production.

2 Resonance Production

We consider the production of an unstable particle d in the reaction

$$a + b \rightarrow c + d. \quad (1)$$

We restrict ourselves further to a specific channel where d decays into two particles e and f :

$$d \rightarrow e + f. \quad (2)$$

In the event that d is a stable particle, the center of mass energy square of the reaction, $\mathfrak{s}_{ab}(= (p_a + p_b)^2)$, is

$$\mathfrak{s}_{ab} = (p_c + p_d)^2. \quad (3)$$

If d is an unstable particle with the attributes of a complex mass $\sqrt{\mathfrak{s}_d}$,

$$\sqrt{\mathfrak{s}_d} = M_R - i\frac{\Gamma_R}{2}, \quad (4)$$

and real 4-velocity \hat{p}_d , then the momenta is complex $p_d = \sqrt{\mathfrak{s}_d}\hat{p}_d$. Therefore, (3) is not satisfied. To find out the result corresponding to (3) in the complex mass case, we consider the bra-ket involving directly the variables p_c , p_d and \mathfrak{s}_{ab} :

$$\langle [d][c]^- | [ab]^+ \rangle \equiv \langle \hat{\mathbf{p}}_d \sigma_d [\mathfrak{s}_d j_d] \eta_{ef} n_{ef}, \hat{\mathbf{p}}_c \sigma_c [m_c j_c]^- | \hat{\mathbf{p}}_{ab} \sigma_{ab} [\mathfrak{s}_{ab} j_{ab}] \eta_{ab} n_{ab}^+ \rangle. \quad (5)$$

In (5),

- $|[ab]^+\rangle \equiv |\hat{\mathbf{p}}_{ab}\sigma_{ab}[\mathbf{s}_{ab}j_{ab}]\eta_{ab}n_{ab}^+\rangle$ is a 4-velocity basis vector for the in-states of the a and b particles, diagonal in the total mass-square \mathbf{s}_{ab} and angular momentum j_{ab} of the $[ab]$ ¹ system [4]. It results from the reduction of the direct product space of the a and b particles, $[a] \otimes [b]$, into a direct sum of irreducible representation spaces [5, 6, 7]. It is labeled by the space component $\hat{\mathbf{p}}_{ab}$ of the 4-velocity $\hat{p}_{ab} = (p_a + p_b)/\sqrt{\mathbf{s}_{ab}}$, the three-component σ_{ab} of j_{ab} , degeneracy quantum numbers η_{ab} that consist of the orbital angular momentum l_{ab} and the total spin s_{ab} of the $[ab]$ system: $\eta_{ab} = \{l_{ab}, s_{ab}\}$, and particle species label n_{ab} designating the masses and spins of a and b : $n_{ab} = \{m_a, s_a, m_b, s_b\}$.
- $|[c]^-\rangle \equiv |\hat{\mathbf{p}}_c\sigma_c[m_cj_c]^-\rangle$ is a 4-velocity basis vector for the c particle.
- $|[d]^-\rangle \equiv |\hat{\mathbf{p}}_d\sigma_d[\mathbf{s}_dj_d]\eta_{ef}n_{ef}^-\rangle$ is a relativistic Gamow vector associated with the $[ef]$ decay channel. It is characterized by a complex mass square \mathbf{s}_d , and real 4-velocity \hat{p}_d .

If d is a stable particle, (5) is an S -matrix element for the transition $ab \rightarrow cd$ for a certain partial-wave of ab . In this case, (5) is proportional to the delta functions giving rise to the conservation condition (3):

$$\begin{aligned} \langle \hat{\mathbf{p}}_d\sigma_d[\mathbf{s}_dj_d], \hat{\mathbf{p}}_c\sigma_c[m_cj_c]^-\rangle & \langle \hat{\mathbf{p}}_{ab}\sigma_{ab}[\mathbf{s}_{ab}j_{ab}]\eta_{ab}n_{ab}^+\rangle = \\ \delta(\mathbf{p}_{ab} - \mathbf{p}_c - \mathbf{p}_d)\delta(\mathbf{s}_{ab} - (p_d + p_c)^2) & \langle \langle \hat{\mathbf{p}}_d\sigma_d[\mathbf{s}_dj_d], \hat{\mathbf{p}}_c\sigma_c[m_cj_c]^-\rangle | S | \hat{\mathbf{p}}_{ab}\sigma_{ab}[\mathbf{s}_{ab}j_{ab}]\eta_{ab}n_{ab}^+\rangle \rangle. \end{aligned} \quad (6)$$

In (6), $\langle \langle \dots | S | \dots \rangle \rangle$ is a reduced S -matrix element resulting from the factoring out of the conservation delta-functions. We calculate how the right side of (6) gets modified in the event that d is unstable. For this calculation, we use for d a Gamow vector.

A calculation of (5) with a Gamow vector is feasible by virtue of the integral representation [2]:

$$|[d]^-\rangle \equiv |\hat{\mathbf{p}}_d\sigma_d[\mathbf{s}_dj_d]\eta_{ef}n_{ef}^-\rangle = \frac{i}{2\pi} \int_{-\infty - i\Gamma}^{\infty} ds'' \frac{|\hat{\mathbf{p}}_d\sigma_d[\mathbf{s}''j_d]\eta_{ef}n_{ef}^-\rangle}{s'' - \mathbf{s}_d}. \quad (7)$$

We should mention that the equality in (7) is not postulated, but is rigorously derived [2] in a relativistic Rigged Hilbert Space [8] scattering theory [9, 10]

¹ $[ab]$ denotes an irreducible representation space of the two-particle system, a and b . Likewise, $[cef]$ denotes an irreducible representation space for the three-particle system, c , e and f .

from the definition of an unstable particle as being the quantity associated with the complex poles of the partial S -matrix.

3 Calculation of (5) with d unstable

In order to calculate (5) with the use of the integral representation (7), we need to insert in (5) two complete set of basis vectors. A complete set of basis vectors is a resolution of the identity with the general form:

$$I = \sum |\text{one particle basis}\rangle\langle\text{one particle basis}| + \sum |\text{two particles basis}\rangle\langle\text{two particles basis}| + \dots \quad (8)$$

In (8), the summation is over all particle species labels and the physical ranges of all quantum numbers, with the understanding that the summation refers to an integration with a specific measure for the continuous quantum numbers. The layout for the calculation of (5) is as follows :

$$\begin{aligned} & \langle [d][c]^- | [ab]^+ \rangle \\ &= \sum_{[ef][c]} \langle [d][c]^- | [ef][c]^- \rangle \langle [ef][c]^- | [ab]^+ \rangle \quad (9) \\ &= \sum_{[ef][c]} \sum_{[cef]} \underbrace{\langle [d][c]^- | [ef][c]^- \rangle}_{\text{Integral representation}} \underbrace{\langle [ef][c]^- | [cef]^- \rangle}_{\text{C-G coefficient}} \underbrace{\langle [cef]^- | [ab]^+ \rangle}_{\text{S-matrix element}} \quad (10) \end{aligned}$$

The above equation means that we first, in (9), insert a complete set of out basis vectors. From (8), we retain only the basis vectors with particle species quantum numbers matching those of $\langle [d][c]^- |$. These are the $[ef] \otimes [c]$ basis system which carry the particle species labels n_{ef} and n_c , Then, in (10), we reinsert the identity (8), and in this case we retain the out basis vectors of the $[cef]^{-1}$ system, $|[cef]^- \rangle$, which naturally carries the same particle species labels as in the $|[ef][c]^- \rangle$ kets. In (10), the first term is evaluated using the integral representation (7) of the Gamow vector. The second term is the Clebsh-Gordan coefficient that arises from the change of basis from the direct product basis vectors $|[ef][c]^- \rangle$ into the direct sum basis vectors $|[cef]^- \rangle$ [7]; and the third term is an S -matrix element for the transition $ab \rightarrow cef$, which is diagonal in the total angular momentum and mass-square (rest-mass energy square) of the in-state, s_{ab} .

Upon performing the calculation outlined in (10), the result is:

$$\begin{aligned}
& \langle \hat{\mathbf{p}}_d \sigma_d [\mathbf{s}_d j_d] \eta_{ef} n_{ef}, \hat{\mathbf{p}}_c \sigma_c [m_c j_c]^- | \hat{\mathbf{p}}_{ab} \sigma_{ab} [\mathbf{s}_{ab} j_{ab}] \eta_{ab} n_{ab}^+ \rangle \\
&= \frac{-i}{2\pi} \frac{1}{s'_{ef} - s_d^*} \frac{1}{s_{ab}^{3/2}} \frac{1}{\sqrt{2} \sqrt{s'_{ef}}} N(\mathbf{s}_{ab}, s'_{ef}, m_c^2) 2\hat{p}_{ab}^0 \delta \left(\hat{\mathbf{p}}_{ab} - \frac{\sqrt{s'_{ef}} \hat{\mathbf{p}}_d + \mathbf{p}_c}{\sqrt{s_{ab}}} \right) \frac{x}{x + \hat{\mathbf{p}}_d \cdot \mathbf{p}_c} \\
& \quad \times \sum_{\eta'_o} P(p'_{ef} p_c, \sigma_d \sigma_c, j_{ab} \sigma_{ab} \eta'_o) \langle s'_{ef} j_d \eta'_o \eta_{ef} n_o | S^{j_{ab}}(\mathbf{s}_{ab}) | \eta_{ab} n_{ab} \rangle. \quad (11)
\end{aligned}$$

where

- $s'_{ef} = x^2 = (p_{ab} - p_c)^2$,
- the subscript o in η'_o and n_o refers to the label: cef ,
- $\langle s'_{ef} j_d \eta'_o \eta_{ef} n_o | S^{j_{ab}}(\mathbf{s}_{ab}) | \eta_{ab} n_{ab} \rangle$ is a reduced S -matrix element:

$$\begin{aligned}
& \langle \hat{\mathbf{p}}'_o \sigma'_o [s'_o j'_o] s'_{ef} j'_{ef} \eta'_o \eta'_{ef} n_o^- | \hat{\mathbf{p}}_{ab} \sigma_{ab} [\mathbf{s}_{ab} j_{ab}] \eta_{ab} n_{ab}^+ \rangle = \\
& 2\hat{p}_{ab}^0 \delta(\hat{\mathbf{p}}_{ab} - \hat{\mathbf{p}}'_o) \delta(s_{ab} - s'_o) \delta_{j'_o j_{ab}} \delta_{\sigma'_o \sigma_{ab}} \langle s'_{ef} j'_{ef} \eta'_o \eta'_{ef} n_o | S^{j_{ab}}(\mathbf{s}_{ab}) | \eta_{ab} n_{ab} \rangle, \quad (12)
\end{aligned}$$

- N is a normalization factor, and P is a factor involving rotation matrices and Clebsh-Gordan coefficients [5, 7, 4].

The limit of (11) as \mathbf{s}_d becomes real is found to be:

$$\begin{aligned}
& \langle \hat{\mathbf{p}}_d \sigma_d [\mathbf{s}_d j_d] \eta_{ef} n_{ef}, \hat{\mathbf{p}}_c \sigma_c [m_c j_c]^- | \hat{\mathbf{p}}_{ab} \sigma_{ab} [\mathbf{s}_{ab} j_{ab}] \eta_{ab} n_{ab}^+ \rangle \\
& \stackrel{\Gamma_d \rightarrow 0}{=} \frac{1}{s_{ab}^{3/2}} \frac{1}{\sqrt{2} \sqrt{s_d}} N(\mathbf{s}_{ab}, \mathbf{s}_d, m_c^2) 2\hat{p}_{ab}^0 \delta(\hat{\mathbf{p}}_{ab} - \hat{\mathbf{p}}'_o) \frac{x}{x + \hat{\mathbf{p}}_d \cdot \mathbf{p}_c} \delta(s_d - (p_{ab} - p_c)^2) \\
& \quad \times \sum_{\eta'_o} P(p_d p_c, \sigma_d \sigma_c, j_{ab} \sigma_{ab} \eta'_o) \langle \mathbf{s}_d j_d \eta'_o \eta_{ef} n_o | S^{j_{ab}}(\mathbf{s}_{ab}) | \eta_{ab} n_{ab} \rangle. \quad (13)
\end{aligned}$$

where

$$\begin{aligned}
\hat{\mathbf{p}}'_o &= \frac{\sqrt{s_d} \hat{\mathbf{p}}_d + \mathbf{p}_c}{\sqrt{s_{ab}}} \\
s'_o &= (p_d + p_c)^2. \quad (14)
\end{aligned}$$

4 Summary and Discussion

We observe that the passage from real \mathfrak{s}_d (13) to a complex \mathfrak{s}_d (11) occurs with the following changes:

- The mass-square condition $\mathfrak{s}_d = (p_{ab} - p_c)^2$ (3) expressed in (13) by the delta function:

$$\delta(\mathfrak{s}_d - (p_{ab} - p_c)^2) \quad (15)$$

gets replaced in the unstable case (11) by a Breit-Wigner distribution in $(p_{ab} - p_c)^2$:

$$\frac{-i}{2\pi} \frac{1}{\mathfrak{s}'_{ef} - \mathfrak{s}_d^*} = \frac{-i}{2\pi} \frac{1}{(p_{ab} - p_c)^2 - (M_R + i\Gamma_R)^2}. \quad (16)$$

This calculational result can be traced to the fact that an unstable particle, unlike a stable one, does not have a definite (real) mass, rather is the superposition of the *whole* spectrum of masses with Breit-Wigner weights, as expressed in (7).

- The condition on the space-components of the 4-velocities in the stable case (13):

$$\hat{\mathbf{p}}_{ab} = \frac{\sqrt{\mathfrak{s}_d} \hat{\mathbf{p}}_d + \mathbf{p}_c}{\sqrt{\mathfrak{s}_{ab}}} \quad (17)$$

expressed by the delta function:

$$\delta(\hat{\mathbf{p}}_{ab} - \hat{\mathbf{p}}'_o) \quad (18)$$

gets replaced by:

$$\hat{\mathbf{p}}_{ab} = \frac{\sqrt{(p_{ab} - p_c)^2} \hat{\mathbf{p}}_d + \mathbf{p}_c}{\sqrt{\mathfrak{s}_{ab}}}. \quad (19)$$

Hence, even though \mathfrak{s}_d is complex, $\hat{\mathbf{p}}_{ab}$ remains real as the quantity $\sqrt{\mathfrak{s}'_{ef}} = \sqrt{(p_{ab} - p_c)^2}$ replaces $\sqrt{\mathfrak{s}_d}$ in (13), and $\hat{\mathbf{p}}_d$ is real according to a defining postulate of the relativistic Gamow vectors.

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References

- [1] D. E. Groom et. al., Particle Data Group, The European Physical Journal C 15, 1 (2000).
- [2] A. Bohm, H. Kaldass and S. Wickramasekara, *Time Asymmetric Quantum Theory - II. Relativistic Resonances From S-Matrix Poles*, Fortschritte der Physik **51**, 569 (2003).
- [3] A. Bohm, H. Kaldass and S. Wickramasekara, *Time Asymmetric Quantum Theory - III. Decaying States and the Causal Poincaré Semigroup*, Fortschritte der Physik **51**, 604 (2003).
- [4] Bohm, A. and Kaldass, H., Phys. Rev. A **60**, 4606–4615 (1999).
- [5] A. S. Wightman, in *Relations de Dispersion et Particules Élémentaires*, C De Witt, R. Omnès [Eds.] (Hermann, Paris, 1960), p. 159.
- [6] H. Joos, Fortschr. Physik **10**, 65 (1962).
- [7] A. J. Macfarlane, Rev. Mod. Phys. **34**, 41 (1962).
- [8] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions*, Vol. 4 (Academic Press, New York, 1964); K. Maurin, *Generalized Eigenfunction Expansions and Unitary Representations of Topological Groups* (Polish Scientific Publishers, Warszawa, 1968).
- [9] A. Bohm, Lett. Math. Phy. **3**, 455 (1978); A. Bohm, J. Math. Phys. **22** 2813 (1981). A. Bohm, S. Maxson, M. Loewe and M. Gadella, Physica A **236**, 485 (1997); A. Bohm, *Quantum Mechanics*, third edition, (Springer, 1993), Chapter XXI.
- [10] A. Bohm and M. Gadella, *Dirac Kets, Gamow Vectors, and Gel'fand Triplets*, Lecture Notes in Physics, Vol. 348 (Springer, Berlin, 1989).