

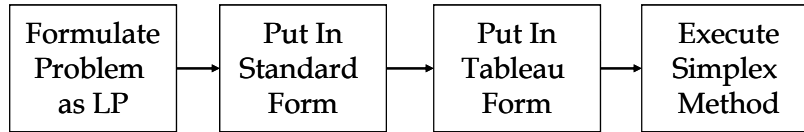
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Linear Programming: The Simplex Method

- An Overview of the Simplex Method
- Standard Form
- Tableau Form
- Setting Up the Initial Simplex Tableau
- Improving the Solution
- Calculating the Next Tableau
- Solving a Minimization Problem
- Special Cases

Overview of the Simplex Method

■ Steps Leading to the Simplex Method



Example: Initial Formulation

■ A Minimization Problem

$$\begin{array}{ll} \text{MIN} & 2x_1 - 3x_2 - 4x_3 \\ \text{s. t.} & x_1 + x_2 + x_3 \leq 30 \\ & 2x_1 + x_2 + 3x_3 \geq 60 \\ & x_1 - x_2 + 2x_3 = 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Standard Form

- An LP is in standard form when:
 - All variables are non-negative
 - All constraints are equalities
- Putting an LP formulation into standard form involves:
 - Adding slack variables to “ \leq ” constraints
 - Subtracting surplus variables from “ \geq ” constraints.

Example: Standard Form

- Problem in Standard Form

$$\begin{array}{ll} \text{MIN} & 2x_1 - 3x_2 - 4x_3 \\ \text{s. t.} & x_1 + x_2 + x_3 + s_1 = 30 \\ & 2x_1 + x_2 + 3x_3 - s_2 = 60 \\ & x_1 - x_2 + 2x_3 = 20 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{array}$$

Tableau Form

- A set of equations is in tableau form if for each equation:
 - its right hand side (RHS) is non-negative, and
 - there is a basic variable. (A basic variable for an equation is a variable whose coefficient in the equation is +1 and whose coefficient in all other equations of the problem is 0.)
- To generate an initial tableau form:
 - An artificial variable must be added to each constraint that does not have a basic variable.

Example: Tableau Form

- Problem in Tableau Form

$$\begin{array}{l} \text{MIN} \quad 2x_1 - 3x_2 - 4x_3 + 0s_1 - 0s_2 + Ma_2 + Ma_3 \\ \text{s. t.} \quad x_1 + x_2 + x_3 + s_1 \qquad \qquad \qquad = 30 \\ \qquad \quad 2x_1 + x_2 + 3x_3 \qquad - s_2 + a_2 \qquad \qquad = 60 \\ \qquad \quad x_1 - x_2 + 2x_3 \qquad \qquad \qquad + a_3 = 20 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad x_1, x_2, x_3, s_1, s_2, a_2, a_3 \geq 0 \end{array}$$

Simplex Tableau

- The simplex tableau is a convenient means for performing the calculations required by the simplex method.

Setting Up Initial Simplex Tableau

- Step 1: If the problem is a minimization problem, multiply the objective function by -1.
- Step 2: If the problem formulation contains any constraints with negative right-hand sides, multiply each constraint by -1.
- Step 3: Add a slack variable to each \leq constraint.
- Step 4: Subtract a surplus variable and add an artificial variable to each \geq constraint.

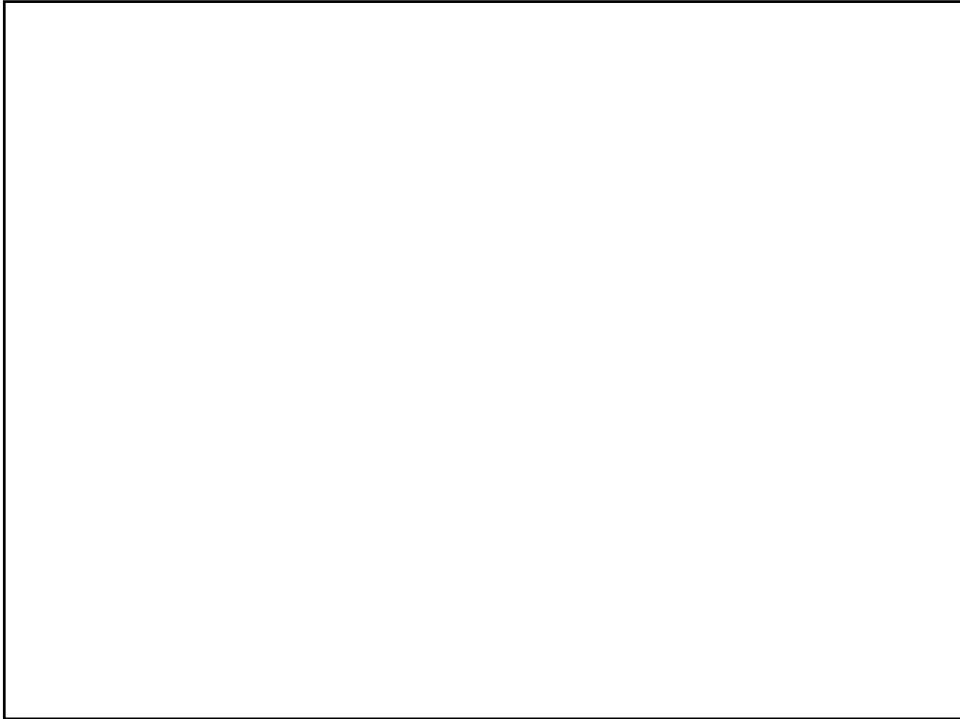
Setting Up Initial Simplex Tableau

- Step 5: Add an artificial variable to each = constraint.
- Step 6: Set each slack and surplus variable's coefficient in the objective function equal to zero.
- Step 7: Set each artificial variable's coefficient in the objective function equal to $-M$, where M is a very large number.
- Step 8: Each slack and artificial variable becomes one of the basic variables in the initial basic feasible solution.

Example: Simplex Method

- Solve the following problem by the simplex method:

$$\begin{array}{ll} \text{Max} & 12x_1 + 18x_2 + 10x_3 \\ \text{s.t.} & 2x_1 + 3x_2 + 4x_3 \leq 50 \\ & x_1 - x_2 - x_3 \geq 0 \\ & x_2 - 1.5x_3 \geq 0 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$



■ Example: Initial Simplex Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	2	3	4	1	0	0	50
s_2	0	-1	1	1	0	1	0	0
s_3	0	0	-1	1.5	0	0	1	0
z_j		0	0	0	0	0	0	0
$c_j - z_j$		12	18	10	0	0	0	

Simplex Method

- Step 1: Determine Entering Variable
 - Identify the variable with the most positive value in the $c_j - z_j$ row. (The entering column is called the pivot column.)
- Step 2: Determine Leaving Variable
 - For each positive number in the entering column, compute the ratio of the right-hand side values divided by these entering column values.
 - If there are no positive values in the entering column, STOP; the problem is unbounded.
 - Otherwise, select the variable with the minimal ratio. (The leaving row is called the pivot row.)

■ Example: Initial Simplex Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	2	3	4	1	0	0	50
s_2	0	-1	1	1	0	1	0	0
s_3	0	0	-1	1.5	0	0	1	0
z_j		0	0	0	0	0	0	0
$c_j - z_j$		12	18	10	0	0	0	

Simplex Method

- Step 3: Generate Next Tableau
 - Divide the pivot row by the pivot element (the entry at the intersection of the pivot row and pivot column) to get a new row. We denote this new row as (row *).
 - Replace each non-pivot row i with:

$$[\text{new row } i] = [\text{current row } i] - [(a_{ij}) \times (\text{row } *)],$$
 where a_{ij} is the value in entering column j of row i

■ Example: Initial Simplex Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	2	3	4	1	0	0	50
s_2	0	-1	1	1	0	1	0	0
s_3	0	0	-1	1.5	0	0	1	0
z_j		0	0	0	0	0	0	0
$c_j - z_j$		12	18	10	0	0	0	

Simplex Method

- Step 4: Calculate z_j Row for New Tableau
 - For each column j , multiply the objective function coefficients of the basic variables by the corresponding numbers in column j and sum them.

Simplex Method

- Step 5: Calculate $c_j - z_j$ Row for New Tableau
 - For each column j , subtract the z_j row from the c_j row.
 - If none of the values in the $c_j - z_j$ row are positive, GO TO STEP 1.
 - If there is an artificial variable in the basis with a positive value, the problem is infeasible. STOP.
 - Otherwise, an optimal solution has been found. The current values of the basic variables are optimal. The optimal values of the non-basic variables are all zero.
 - If any non-basic variable's $c_j - z_j$ value is 0, alternate optimal solutions might exist. STOP.

Example: Simplex Method

- Solve the following problem by the simplex method:

$$\begin{aligned} \text{Max} \quad & 12x_1 + 18x_2 + 10x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 4x_3 \leq 50 \\ & x_1 - x_2 - x_3 \geq 0 \\ & x_2 - 1.5x_3 \geq 0 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Example: Simplex Method

- Writing the Problem in Tableau Form

We can avoid introducing artificial variables to the second and third constraints by multiplying each by -1 (making them \leq constraints). Thus, slack variables s_1 , s_2 , and s_3 are added to the three constraints.

$$\begin{aligned} \text{Max} \quad & 12x_1 + 18x_2 + 10x_3 + 0s_1 + 0s_2 + 0s_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 4x_3 + s_1 = 50 \\ & -x_1 + x_2 + x_3 + s_2 = 0 \\ & -x_2 + 1.5x_3 + s_3 = 0 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Example: Simplex Method

■ Initial Simplex Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	2	3	4	1	0	0	50
s_2	0	-1	1	1	0	1	0	0
s_3	0	0	-1	1.5	0	0	1	0
z_j		0	0	0	0	0	0	0
$c_j - z_j$		12	18	10	0	0	0	

Example: Simplex Method

■ Iteration 1

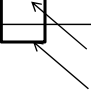
• Step 1: Determine the Entering Variable

The most positive $c_j - z_j = 18$. Thus x_2 is the entering variable.

Example: Simplex Method

■ Initial Simplex Tableau (entering variable x_2)

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	2	3	4	1	0	0	50
s_2	0	-1	1	1	0	1	0	0
s_3	0	0	-1	1.5	0	0	1	0
z_j		0	0	0	0	0	0	0
$c_j - z_j$		12	18	10	0	0	0	



 Max.
 Pivot column

Example: Simplex Method

■ Iteration 1

• Step 2: Determine the Leaving Variable

Take the ratio between the right hand side and positive numbers in the x_2 column:

$$50/3 = 16 \frac{2}{3}$$

$$0/1 = 0 \quad \longleftarrow \quad \text{minimum}$$

s_2 is the leaving variable and the 1 is the pivot element.

Example: Simplex Method

■ Initial Simplex Tableau (leaving variable s_2)

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	2	3	4	1	0	0	50
s_2	0	-1	1	1	0	1	0	0
s_3	0	0	-1	1.5	0	0	1	0
z_j		0	0	0	0	0	0	0
$c_j - z_j$		12	18	10	0	0	0	

Min (RHS/C
pivot column

Example: Simplex Method

■ Iteration 1 (continued)

• Step 3: Generate New Tableau

Divide the second row by 1, the pivot element.
Call the "new" (in this case, unchanged) row the "*
row".

make coefficient for other constraints = 0

Example: Simplex Method

■ Initial Simplex Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	2	3	4	1	0	0	50
s_2	0	-1	①	1	0	1	0	0 (* row)
s_3	0	0	-1	1.5	0	0	1	0
z_j		0	0	0	0	0	0	0
$c_j - z_j$		12	18	10	0	0	0	

Example: Simplex Method

■ Iteration 1 (continued)

- Step 3: Generate New Tableau .

Subtract 3 x (* row) from row 1.

Subtract -1 x (* row) from row 3.

New rows 1, 2, and 3 are shown in the upcoming tableau.

Example: Simplex Method

- Iteration 1 (continued) - New Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	5	0	1	1	-3	0	50
x_2	18	-1	1	1	0	1	0	0
s_3	0	-1	0	2.5	0	1	1	0
z_j		-18	18	18	0	18	0	0
$c_j - z_j$		30	0	-8	0	-18	0	

Example: Simplex Method

- Iteration 1 (continued)

- Step 4: Calculate z_j Row for New Tableau

The new z_j row values are obtained by multiplying the c_B column by each column, element by element and summing.

For example, $z_1 = 5(0) + -1(18) + -1(0) = -18$.

Example: Simplex Method

- Iteration 1 (continued)

- Step 5: Calculate $c_j - z_j$ Row for New Tableau

The new $c_j - z_j$ row values are obtained by subtracting z_j value in a column from the c_j value in the same column.

For example, $c_1 - z_1 = 12 - (-18) = 30$.

Example: Simplex Method

- Iteration 1 (continued) - New Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	5	0	1	1	-3	0	50
x_2	18	-1	1	1	0	1	0	0
s_3	0	-1	0	2.5	0	1	1	0
z_j		-18	18	18	0	18	0	0
$c_j - z_j$		30	0	-8	0	-18	0	

Example: Simplex Method

■ Iteration 2

- Step 1: Determine the Entering Variable

The most positive $c_j - z_j = 30$. x_1 is the entering variable.

- Step 2: Determine the Leaving Variable

Take the ratio between the right hand side and positive numbers in the x_1 column:

$$50/5 = 10 \leftarrow \text{minimum}$$

There are no ratios for the second and third rows because their column elements (-1) are negative.

Thus, s_1 (corresponding to row 1) is the leaving variable and 5 is the pivot element.

Example: Simplex Method

■ Iteration 1 (continued) - New Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1	0	5	0	1	1	-3	0	50
x_2	18	-1	1	1	0	1	0	0
s_3	0	-1	0	2.5	0	1	1	0
z_j		-18	18	18	0	18	0	0
$c_j - z_j$		30	0	-8	0	-18	0	

Example: Simplex Method

■ Iteration 1 (continued) - New Tableau

		x_1	x_2	x_3	s_1	s_2	s_3		
Basis	c_B	12	18	10	0	0	0		
s_1	0	1	0	.2	.2	-.6	0	10	(* row)
x_2	18	-1	1	1	0	1	0	0	
s_3	0	-1	0	2.5	0	1	1	0	
z_j		-18	18	18	0	18	0	0	
$c_j - z_j$		30	0	-8	0	-18	0		

Example: Simplex Method

■ Iteration 2 (continued)

• Step 3: Generate New Tableau

Divide row 1 by 5, the pivot element. (Call this new row 1 the "* row").

Subtract $(-1) \times$ (* row) from the second row.

Subtract $(-1) \times$ (* row) from the third row.

• Step 4: Calculate z_j Row for New Tableau

The new z_j row values are obtained by multiplying the c_B column by each column, element by element and summing.

For example, $z_3 = .2(12) + 1.2(18) + .2(0) = 24$.

Example: Simplex Method

■ Iteration 2 (continued)

- Step 5: Calculate $c_j - z_j$ Row for New Tableau

The new $c_j - z_j$ row values are obtained by subtracting z_j value in a column from the c_j value in the same column.

For example, $c_3 - z_3 = 10 - (24) = -14$.

Since there are no positive numbers in the $c_j - z_j$ row, this tableau is optimal. The optimal solution is: $x_1 = 10$; $x_2 = 10$; $x_3 = 0$; $s_1 = 0$; $s_2 = 0$ $s_3 = 10$, and the optimal value of the objective function is 300.

Example: Simplex Method

■ Iteration 2 (continued) – Final Tableau

		x_1	x_2	x_3	s_1	s_2	s_3		
Basis	c_B	12	18	10	0	0	0		
x_1	12	1	0	.2	.2	-.6	0	10	(* row)
x_2	18	0	1	1.2	.2	.4	0	10	
s_3	0	0	0	2.7	.2	.4	1	10	
z_j		12	18	24	6	0	0	300	
$c_j - z_j$		0	0	-14	-6	0	0		

Special Cases

- Infeasibility
- Unboundedness
- Alternative Optimal Solution

Infeasibility

- Infeasibility is detected in the simplex method when an artificial variable remains positive in the final tableau.

Example: Infeasibility

■ LP Formulation

$$\begin{aligned}
 \text{MAX } & 2x_1 + 6x_2 \\
 \text{s. t. } & 4x_1 + 3x_2 \leq 12 \\
 & 2x_1 + x_2 \geq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

Example: Infeasibility

■ Final Tableau

		x_1	x_2	s_1	s_2	a_2	
Basis	C_B	2	6	0	0	$-M$	
x_1	2	1	$3/4$	$1/4$	0	0	3
a_2	$-M$	0	$-1/2$	$-1/2$	-1	1	2
	z_j	2	$(1/2)M$ $+3/2$	$(1/2)M$ $+1/2$	M	$-M$	$-2M$ $+6$
	$c_j - z_j$	0	$-(1/2)M$ $+9/2$	$-(1/2)M$ $-1/2$	$-M$	0	

Example: Infeasibility

In the previous slide we see that the tableau is the final tableau because all $c_j - z_j \leq 0$. However, an artificial variable is still positive, so the problem is infeasible.

Unboundedness

- A linear program has an unbounded solution if all entries in an entering column are non-positive.

Example: Unboundedness

■ LP Formulation

$$\begin{aligned}
 \text{MAX} \quad & 2x_1 + 6x_2 \\
 \text{s. t.} \quad & 4x_1 + 3x_2 \geq 12 \\
 & 2x_1 + x_2 \geq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

Example: Unboundedness

■ Final Tableau

		x_1	x_2	s_1	s_2	
Basis	c_B	3	4	0	0	
x_2	4	3	1	0	-1	8
s_1	0	2	0	1	-1	3
z_j		12	4	0	-4	32
$c_j - z_j$		-9	0	0	4	

Example: Unboundedness

In the previous slide we see that $c_4 - z_4 = 4$ (is positive), but its column is all non-positive. This indicates that the problem is unbounded.

Alternative Optimal Solution

- A linear program has alternate optimal solutions if the final tableau has a $c_j - z_j$ value equal to 0 for a non-basic variable.

Example: Alternative Optimal Solution

■ Final Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	s_4	
Basis	c_B	2	4	6	0	0	0	0	
s_3	0	0	0	2	4	-2	1	0	8
x_2	4	0	1	2	2	-1	0	0	6
x_1	2	1	0	-1	1	2	0	0	4
s_4	0	0	0	1	3	2	0	1	12
	z_j	2	4	6	10	0	0	0	32
	$c_j - z_j$	0	0	0	-10	0	0	0	

Example: Alternative Optimal Solution

In the previous slide we see that the optimal solution is:

$$x_1 = 4, x_2 = 6, x_3 = 0, \text{ and } z = 32$$

Note that x_3 is non-basic and its $c_3 - z_3 = 0$. This 0 indicates that if x_3 were increased, the value of the objective function would not change.

Another optimal solution can be found by choosing x_3 as the entering variable and performing one iteration of the simplex method. The new tableau on the next slide shows an alternative optimal solution is:

$$x_1 = 7, x_2 = 0, x_3 = 3, \text{ and } z = 32$$

Example: Alternative Optimal Solution

■ New Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	s_4	
Basis	c_B	2	4	6	0	0	0	0	
s_3	0	0	-1	0	2	-1	1	0	2
x_3	6	0	.5	1	1	-.5	0	0	3
x_1	2	1	.5	0	2	1.5	0	0	7
s_4	0	0	-.5	0	2	2.5	0	1	9
z_j		2	4	6	10	0	0	0	32
$c_j - z_j$		0	0	0	-10	0	0	0	