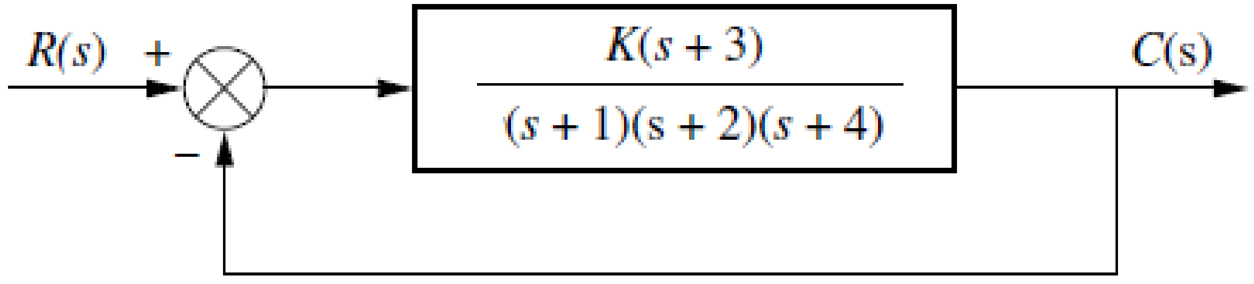
Control Systems I

Lecture 5
Root Locus (cont.)

Emam Fathy
email: emfmz@aast.edu

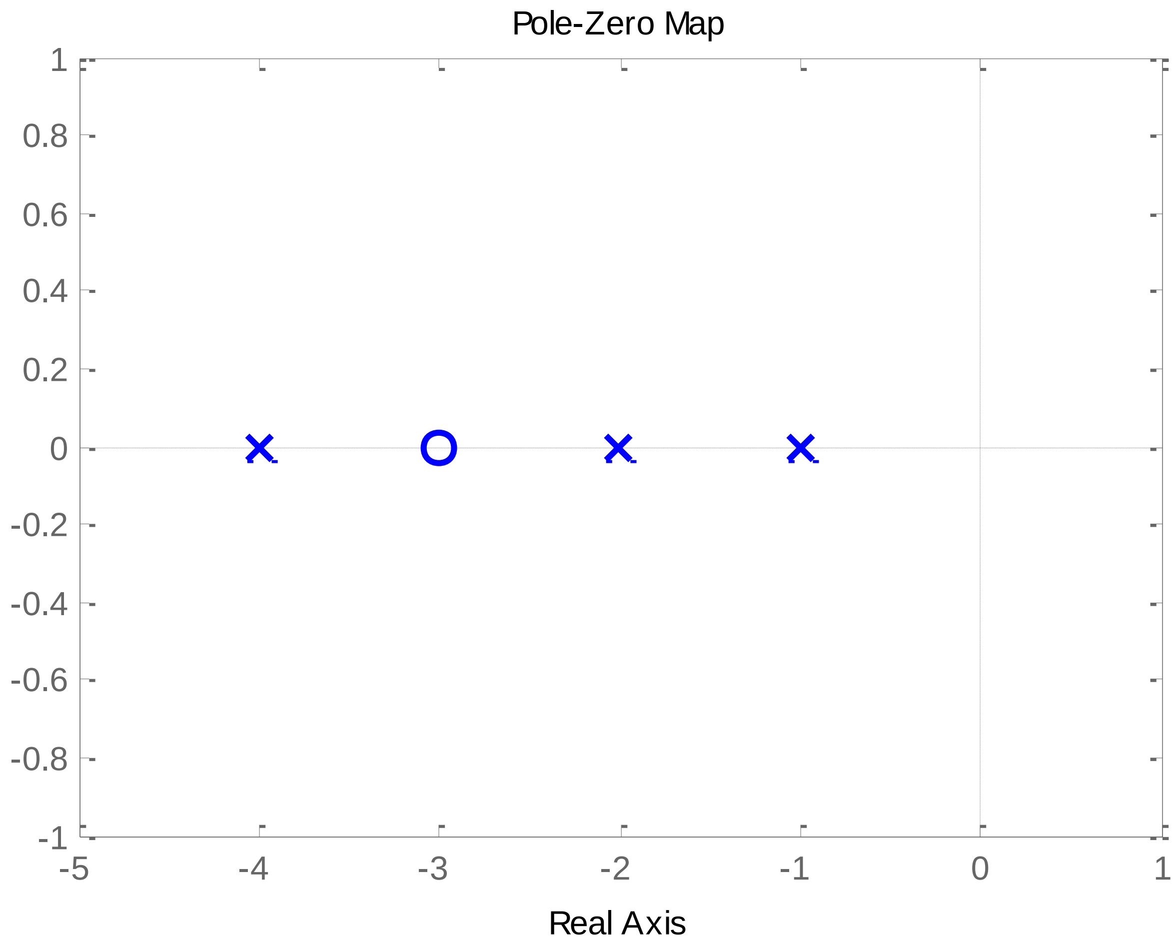
Example

- Sketch the root locus of following system



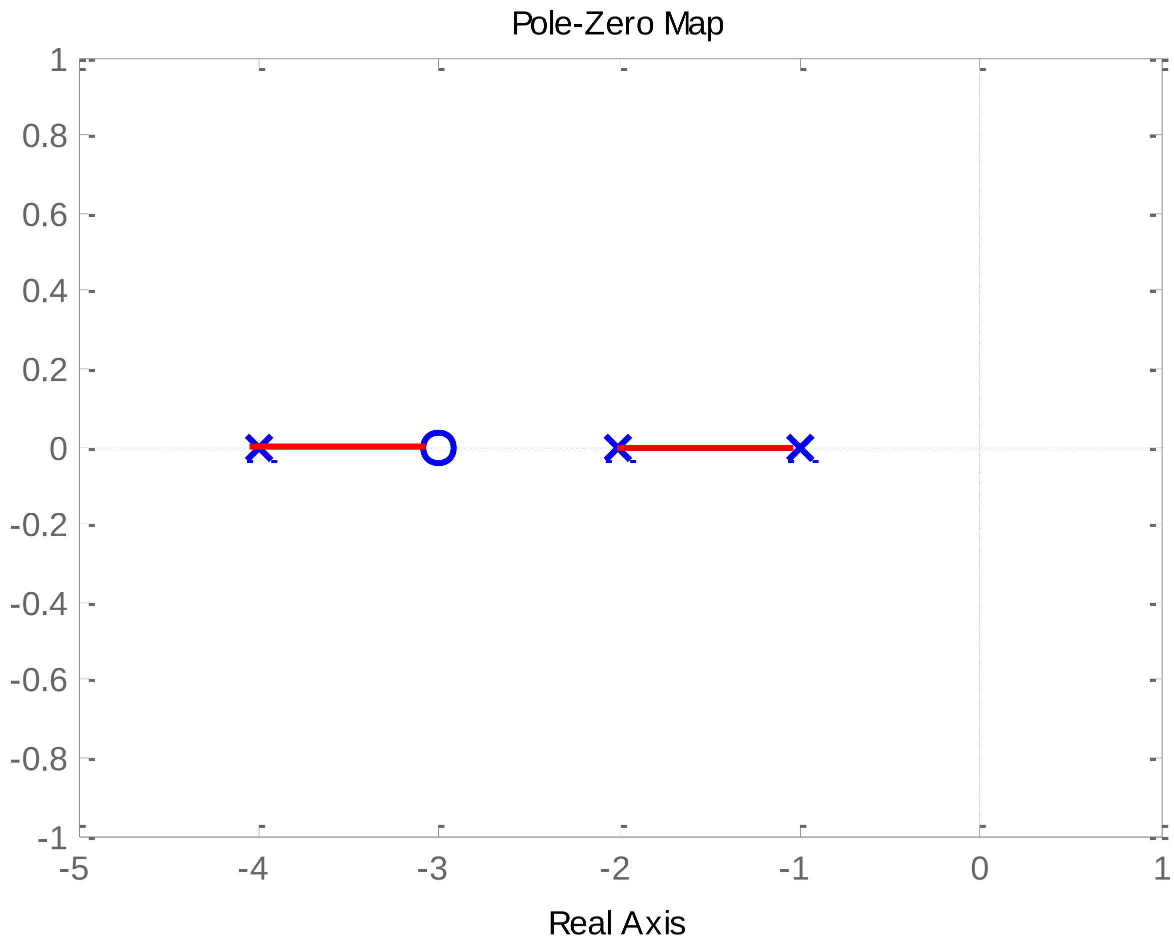
Example#2

- Step-1: Pole-Zero Map



Example#2

- Step-2: Root Loci on Real axis

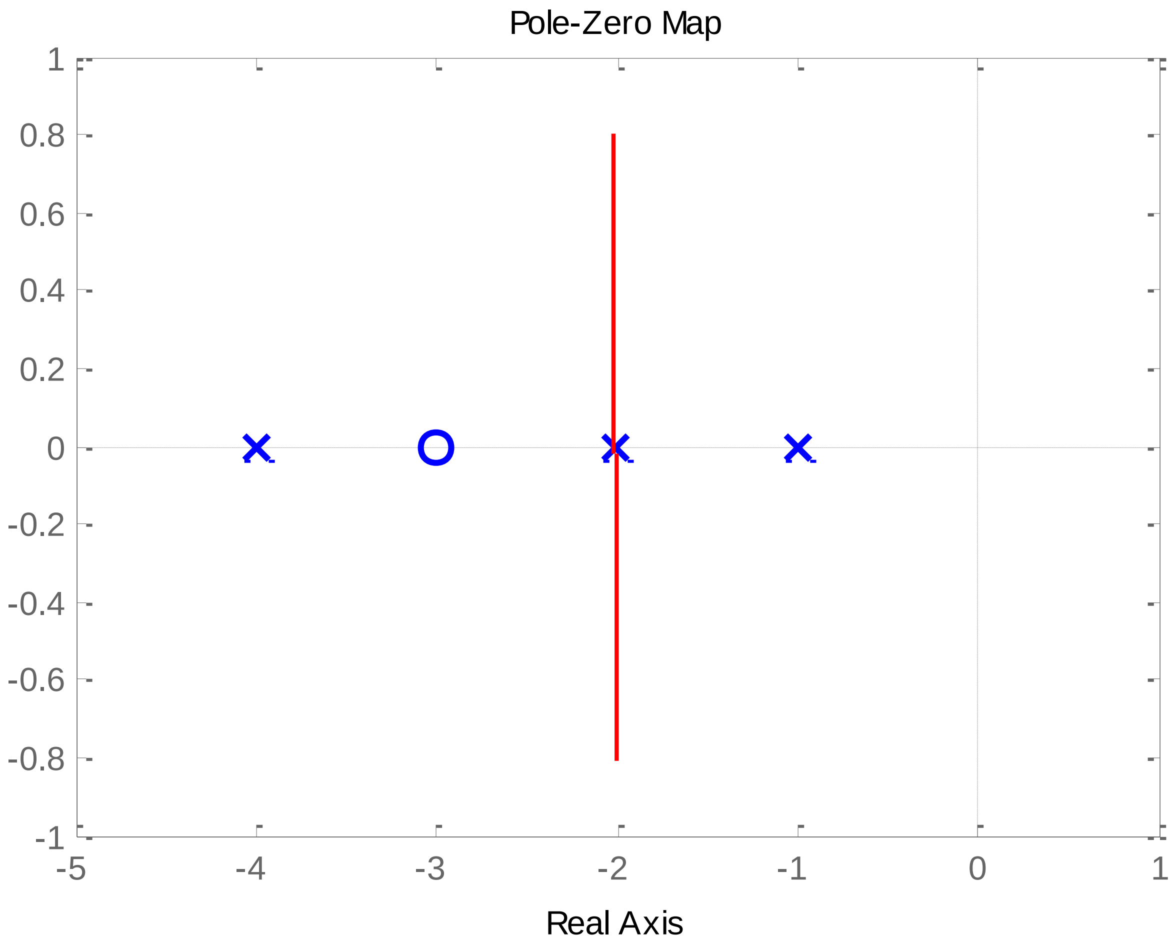


Example#2

- Step-3: Asymptotes

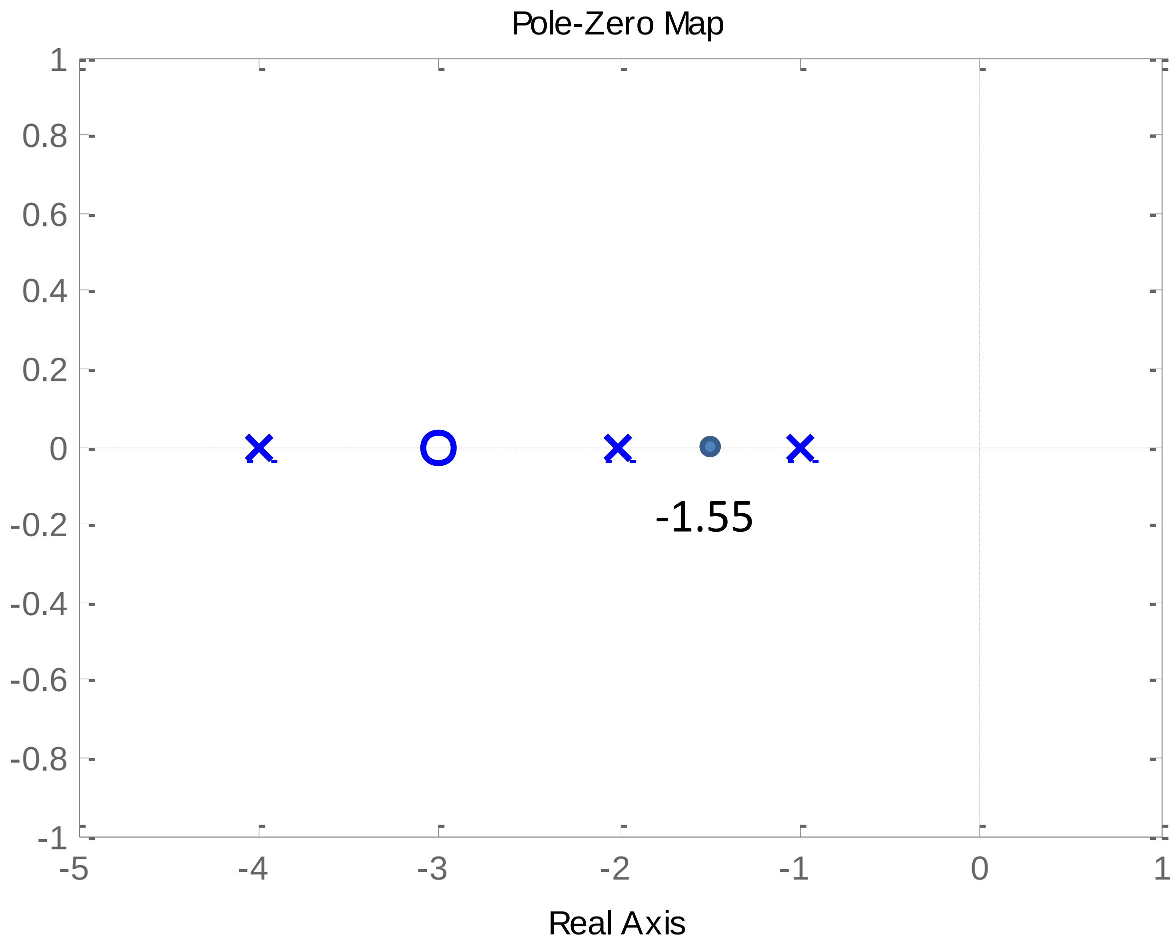
$$\psi = \pm 90°$$

$$\sigma = -2$$

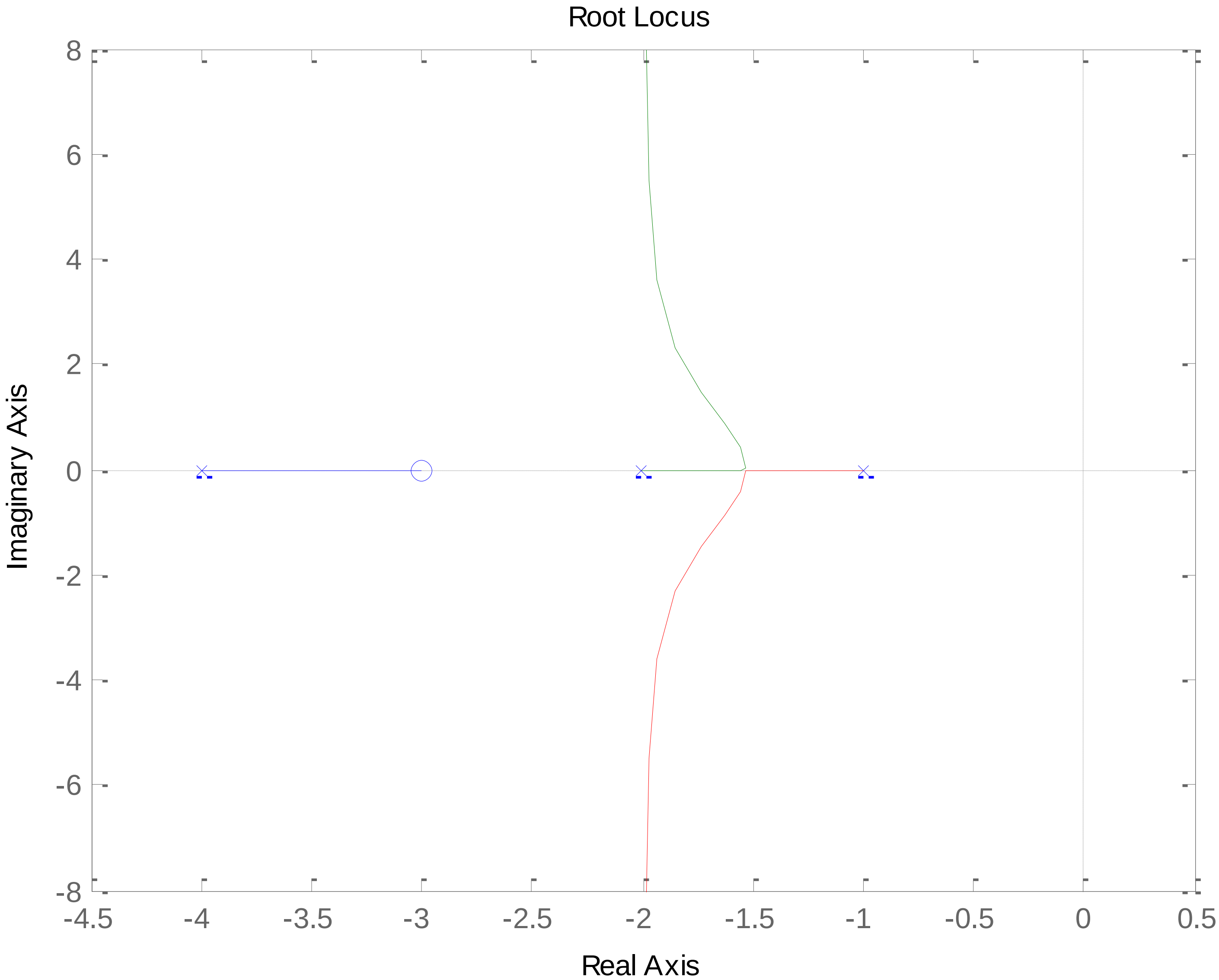


Example#2

- Step-4: breakaway point

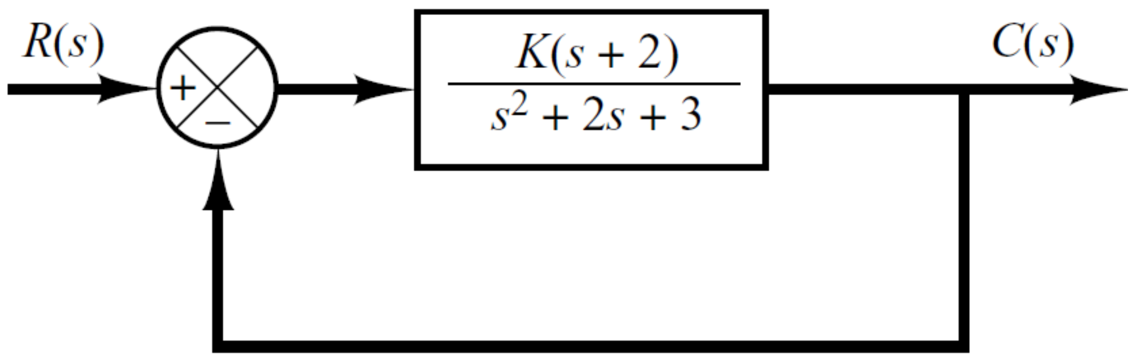


Example#2


Root Locus
Imaginary Axis
Real Axis
8
6
4
2
0
-2
-4
-6
-8
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5

Example

- Sketch the root-locus plot of following system with complex-conjugate open loop poles.



$$G(s) = \frac{K(s+2)}{s^2+2s+3}, \qquad H(s) = 1$$

$G(s)$ has a pair of complex-conjugate poles at

$$s = -1 + j\sqrt{2}, \qquad s = -1 - j\sqrt{2}$$

Example

- *Step-1:* Pole-Zero Map

- *Step-2:* Determine the root loci on real axis

- *Step-3*: Asymptotes

Example

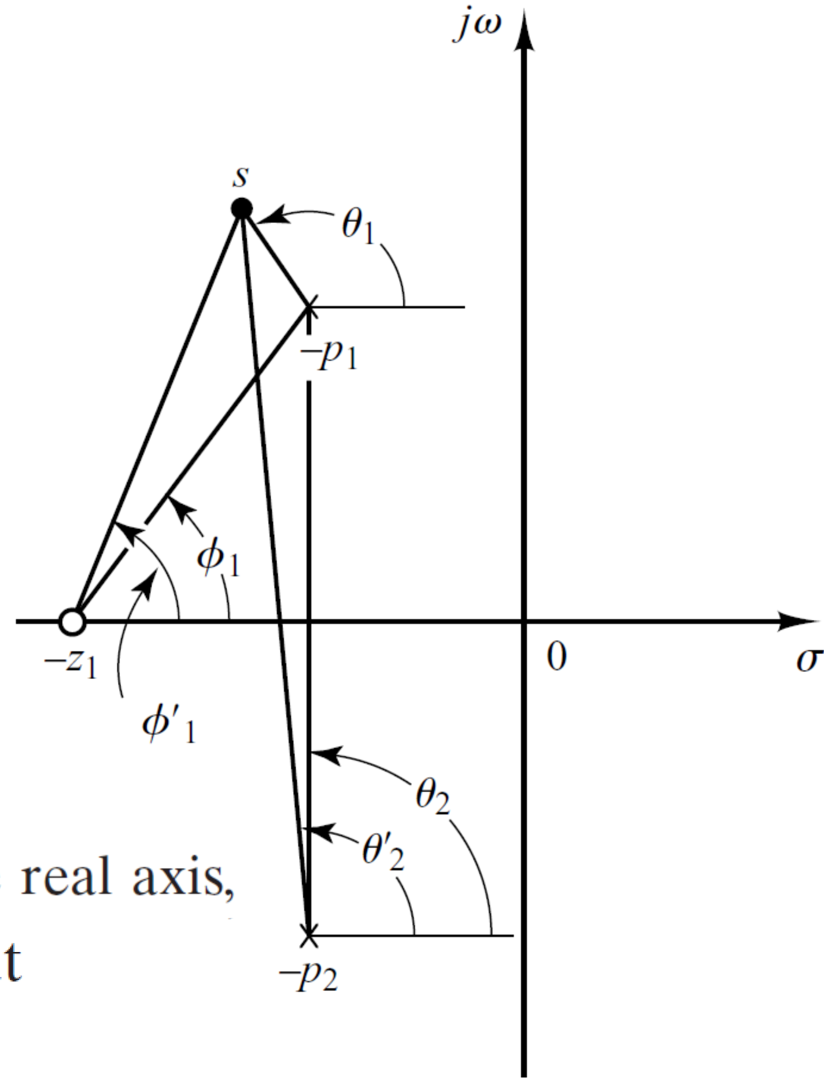
- *<u>Step-4:</u>* Determine the angle of departure from the complex-conjugate open-loop poles.

  – The presence of a pair of complex-conjugate open-loop poles requires the determination of the angle of departure from these poles.

  – Knowledge of this angle is important, since the root locus near a complex pole yields information as to whether the locus originating from the complex pole migrates toward the real axis or extends toward the asymptote.

Example

- *Step-4*: Determine the angle of departure from the complex-conjugate open-loop poles.



The angle of departure is then

$$\theta_1 = 180^\circ - \theta_2 + \phi_1$$
$$= 180^\circ - 90^\circ + 55^\circ = 145^\circ$$

Since the root locus is symmetric about the real axis, the angle of departure from the pole at

$$s = -p_2 \text{ is } -145^\circ$$

Example

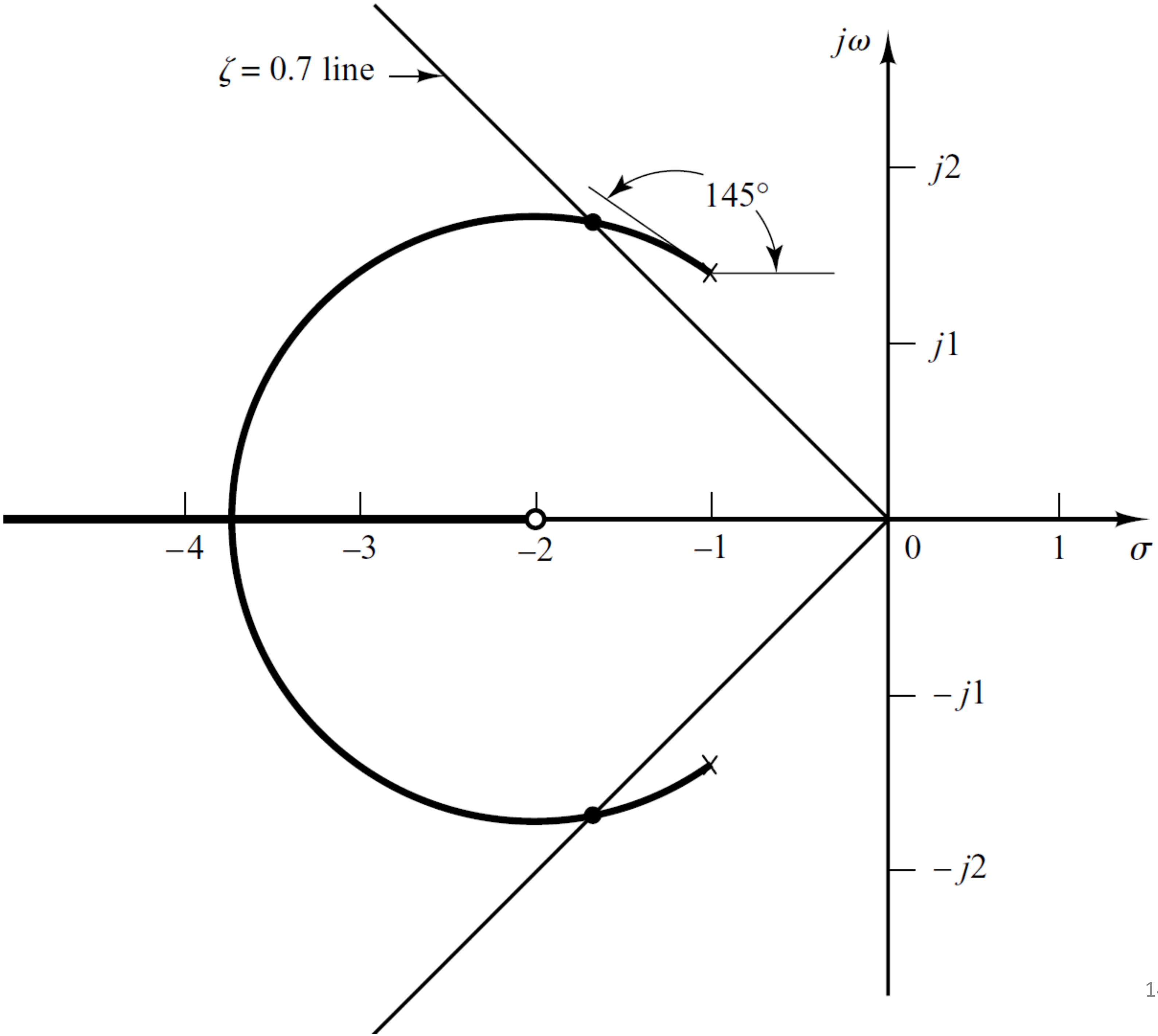
- ***Step-5:*** Break-in point

$$K = -\frac{s^2 + 2s + 3}{s + 2}$$

$$\frac{dK}{ds} = -\frac{(2s + 2)(s + 2) - (s^2 + 2s + 3)}{(s + 2)^2} = 0$$

$$s^2 + 4s + 1 = 0$$

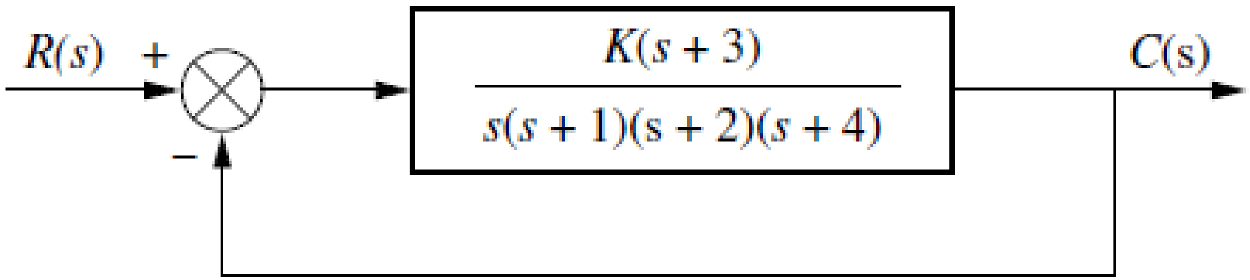
$$s = -3.7320 \qquad \text{or} \qquad s = -0.2680$$



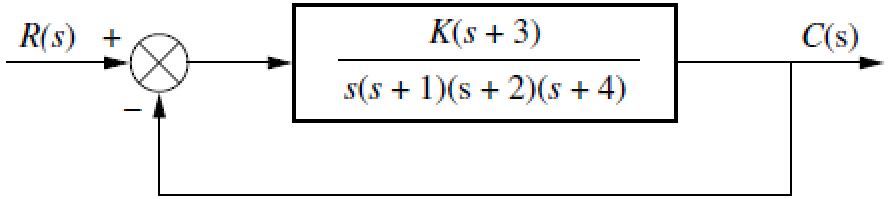
$\zeta = 0.7$ line
$j\omega$
$j2$
$j1$
$-j1$
$-j2$
145°
−4
−3
−2
−1
0
1
σ

Root Locus of Higher Order System

- Sketch the Root Loci of following unity feedback system



$$G(s)H(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$



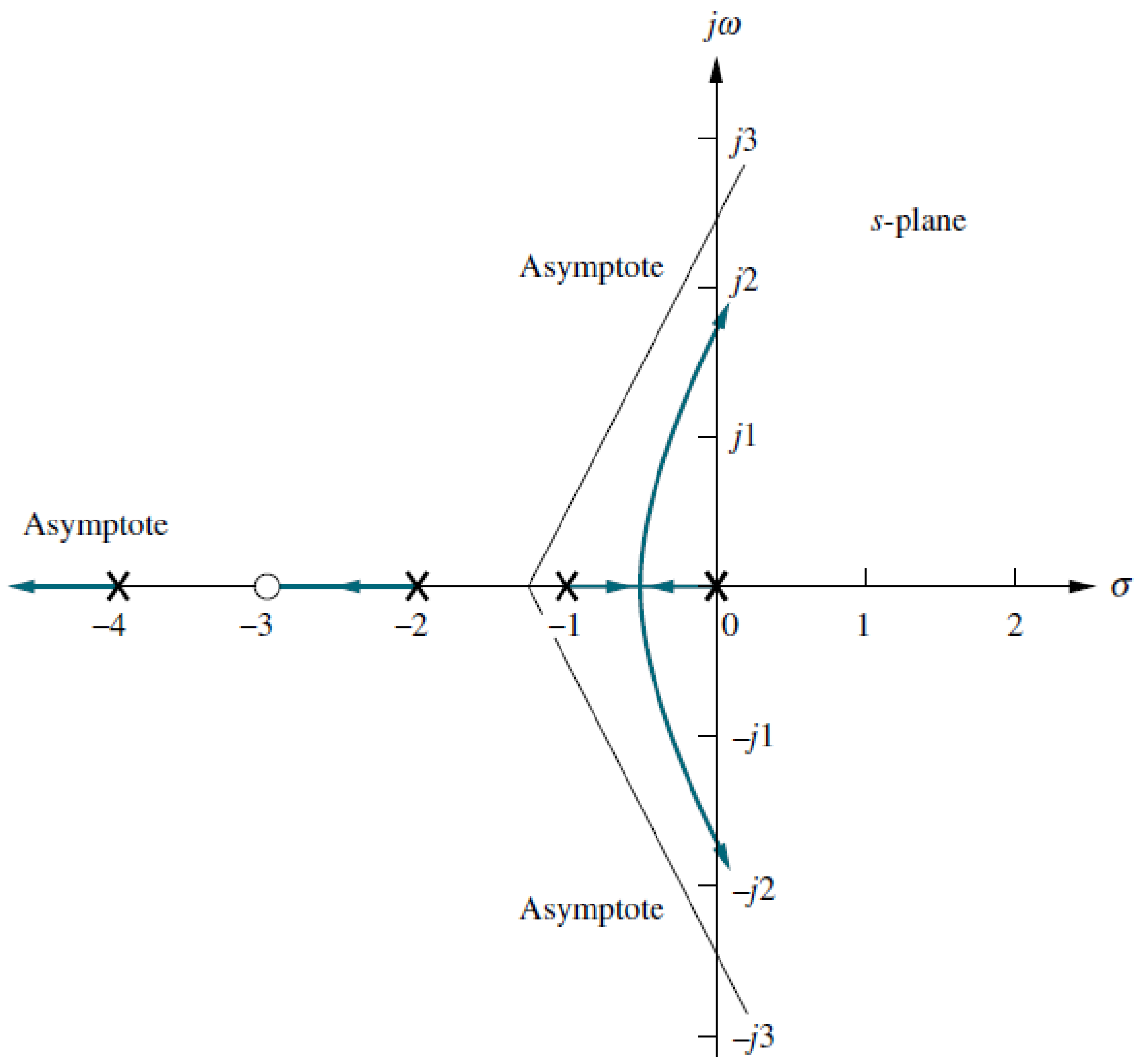
- Let us begin by calculating the asymptotes. The real-axis intercept is evaluated as;

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

- The angles of the lines that intersect at - 4/3, given by

$$\begin{aligned}
\theta_a &= \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \\
&= \pi/3 \qquad \text{for } k = 0 \\
&= \pi \qquad \text{for } k = 1 \\
&= 5\pi/3 \qquad \text{for } k = 2
\end{aligned}$$

- The Figure shows the complete root locus as well as the asymptotes that were just calculated.



Example

$$1 + GH(s) = 1 + \frac{K(s+1)}{s(s+2)(s+4)^2}$$

a) Number of finite poles = n = 4.
b) Number of finite zeros = m = 1.
c) Number of asymptotes = n - m = 3.
d) Number of branches or loci equals to the number of finite poles (n) = 4.
e) The portion of the real-axis between, 0 and -2, and between, -4 and -∞, lie on the root locus for *K* > 0.

- Using Eq. (v), the real-axis asymptotes intercept is evaluated as;

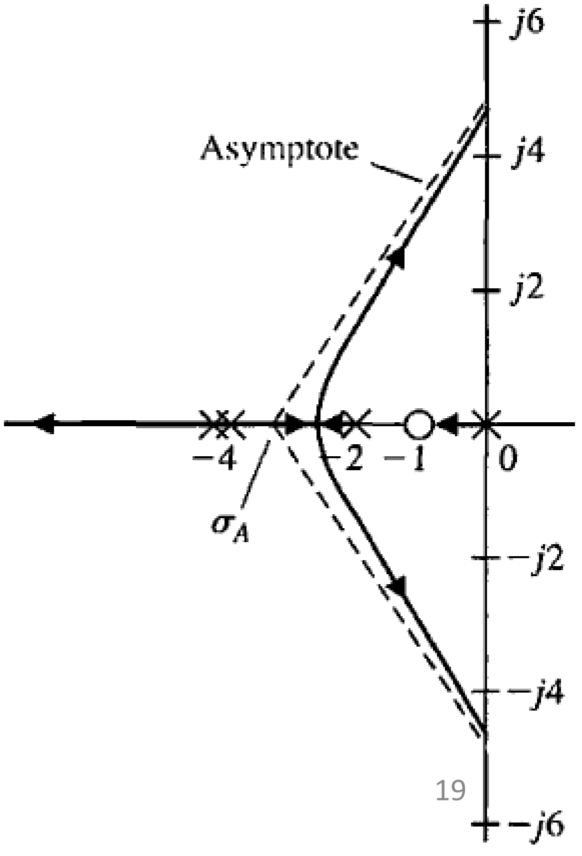
$$\sigma_a = \frac{(-2) + 2(-4) - (-1)}{n - m} = \frac{-10 + 1}{4 - 1} = -3$$

- The angles of the asymptotes that intersect at - 3, given by Eq. (vi), are;

$$\theta_a = \frac{(2k+1)\pi}{n - m} = \frac{(2k+1)\pi}{4 - 1}$$

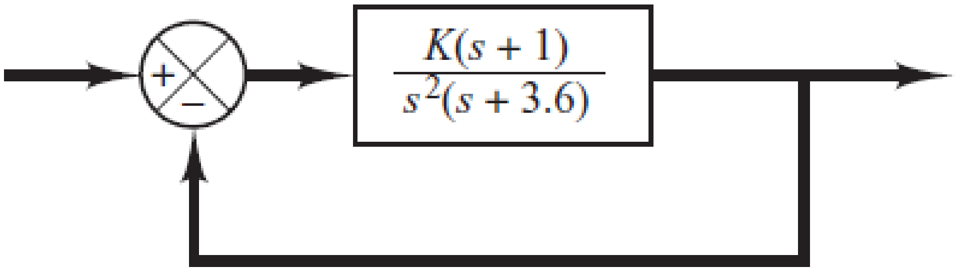
For *K* = 0, $\theta a = 60^o$
For *K* = 1, $\theta a = 180^o$
For *K* = 2, $\theta a = 300^o$

- The root-locus plot of the system is shown in the figure below.
- It is noted that there are three asymptotes. Since n – m = 3.
- The root loci must begin at the poles; two loci (or branches) must leave the double pole at s = -4.
- Using Eq. (vii), the breakaway point, σ, can be determine as;



- The solution of the above equation is $\sigma = -2.59$.

Example



- A root locus exists on the real axis between points ***s* = –1** and ***s* = –3.6**.
- The intersection of the asymptotes and the real axis is determined as,

$$\sigma_a = \frac{0 + 0 + 3.6 - 1}{n - m} = \frac{2.6}{3 - 1} = -1.3$$

- The angles of the asymptotes that intersect at – 1.3, given by Eq. (vi), are;

$$\theta_a = \frac{(2k+1)\pi}{n - m} = \frac{(2k+1)\pi}{3 - 1}$$

For $K = 0$, $\theta a = 90^o$
For $K = 1$, $\theta a = -90^o$ or 270^o

- Since the characteristic equation is $s^3 + 3.6s^2 + K(s + 1) = 0$
- We have $K = -\frac{s^3 + 3.6s^2}{s + 1} \longrightarrow$ (a)

$$K = -\frac{s^3 + 3.6s^2}{s + 1}$$

- The breakaway and break-in points:

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s + 1) - (s^3 + 3.6s^2)}{(s + 1)^2} = 0$$

$$\text{or} \qquad s^3 + 3.3s^2 + 3.6s = 0$$

From which we get,

$$s = 0, \qquad s = -1.65 + j0.9367, \qquad s = -1.65 - j0.9367$$

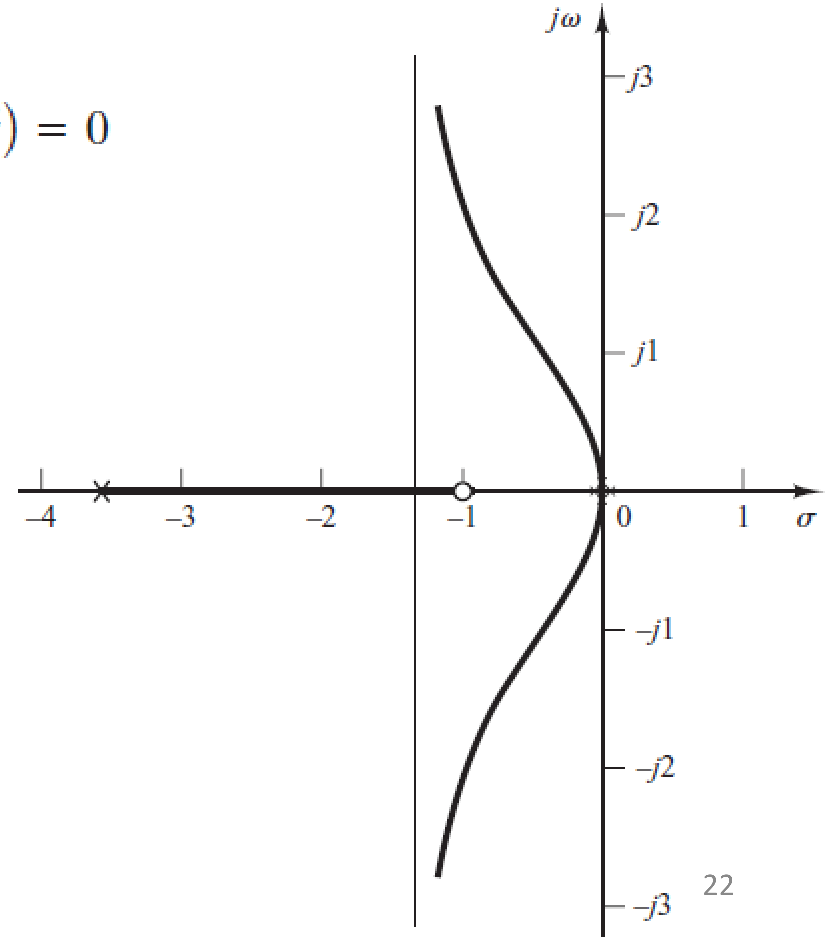
- Point $s = 0$ corresponds to the actual breakaway point. But points $s = 1.65 \pm j0.9367$ neither breakaway nor break-in points, because the corresponding gain values K become complex quantities.

- To check the points where root-locus branches may cross the imaginary axis, substitute $s = j\omega$ into the characteristic equation, yielding.

$$(j\omega)^3 + 3.6(j\omega)^2 + Kj\omega + K = 0$$

or

$$\left(K - 3.6\omega^2\right) + j\omega\left(K - \omega^2\right) = 0$$



- Notice that this equation can be satisfied only if $\omega = 0, K = 0$.
- Because of the presence of a double pole at the origin, the root locus is tangent to the $j\omega$axis at $k = 0$.
- The root-locus branches do not cross the $j\omega$axis.
- The root loci of this system is shown in the Figure.

End of Lec

Note

- Determine the Breakaway and break in points

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

Solution

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)}$$

$$\frac{K(s^2-8s+15)}{s^2+3s+2} = -1$$

$$K = -\frac{(s^2+3s+2)}{(s^2-8s+15)}$$

- Differentiating ***K*** with respect to **s** and setting the derivative equal to zero yields;

$$\frac{dK}{ds} = -\frac{[(s^2-8s+15)(2s+3)-(s^2+3s+2)(2s-8)]}{(s^2-8s+15)^2} = 0$$

$$11s^2 - 26s - 61 = 0$$

Hence, solving for s, we find the break-away and break-in points;

s = -1.45 and 3.82