

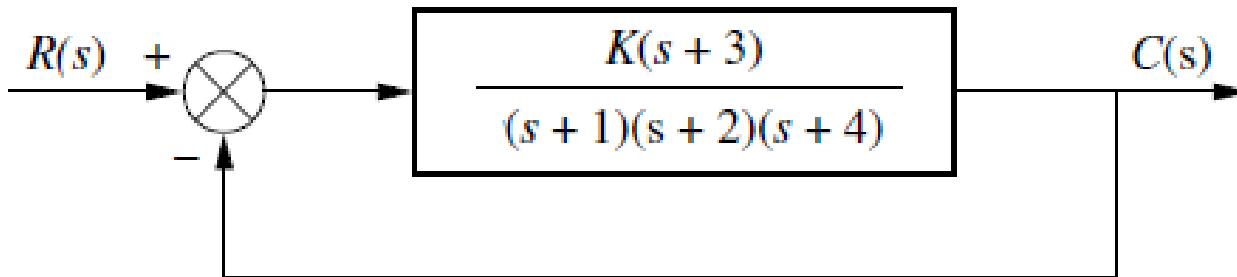
Control Systems I

Lecture 5 Root Locus (cont.)

Emam Fathy
email: emfmz@aast.edu

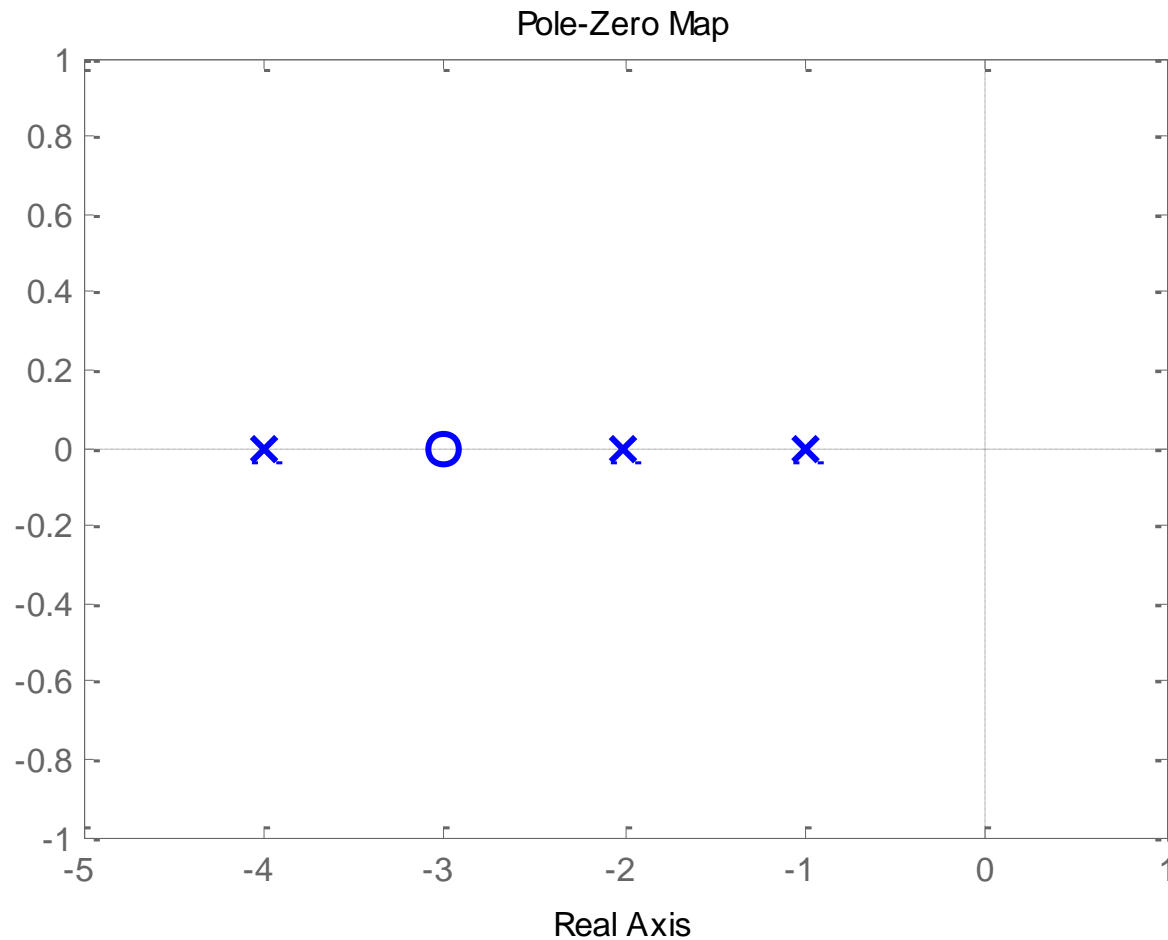
Example

- Sketch the root locus of following system



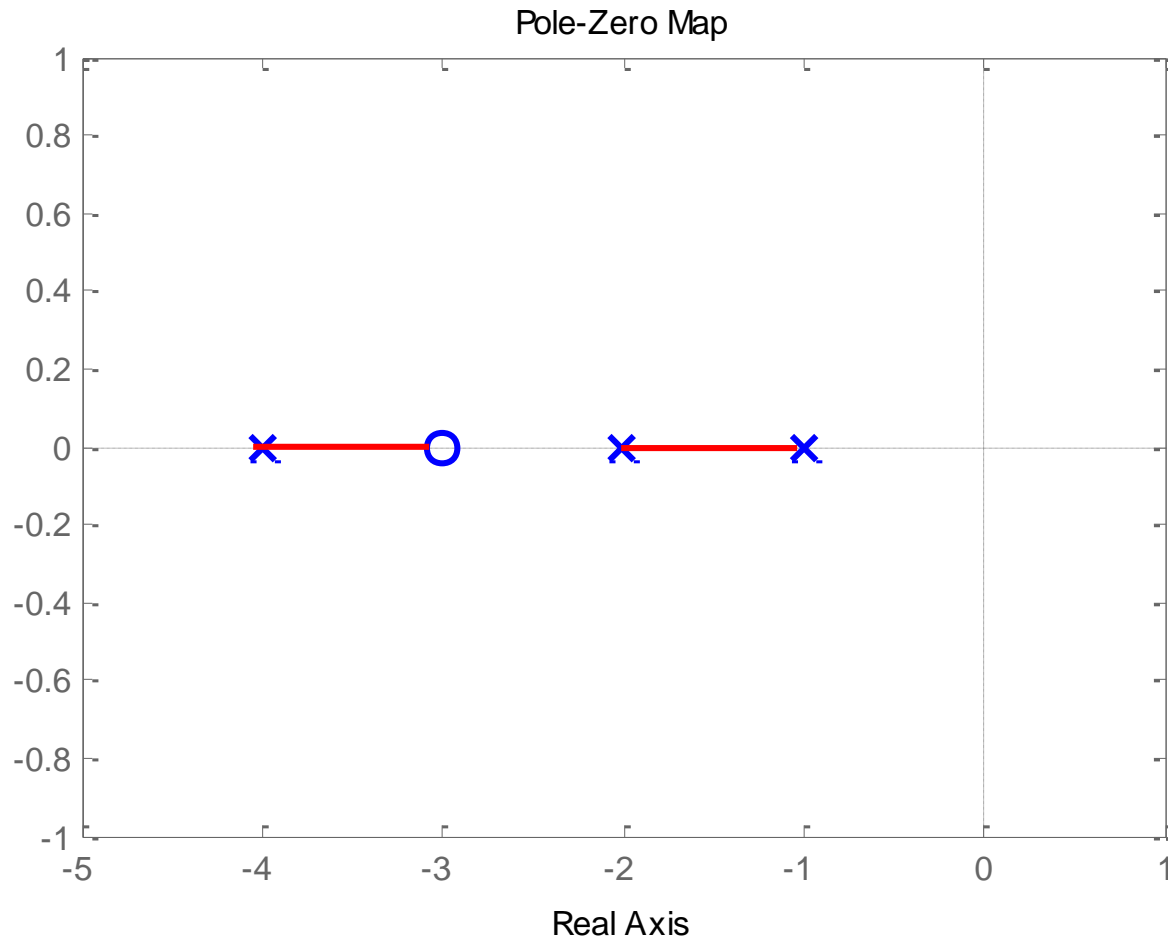
Example#2

- Step-1: Pole-Zero Map



Example#2

- Step-2: Root Loci on Real axis

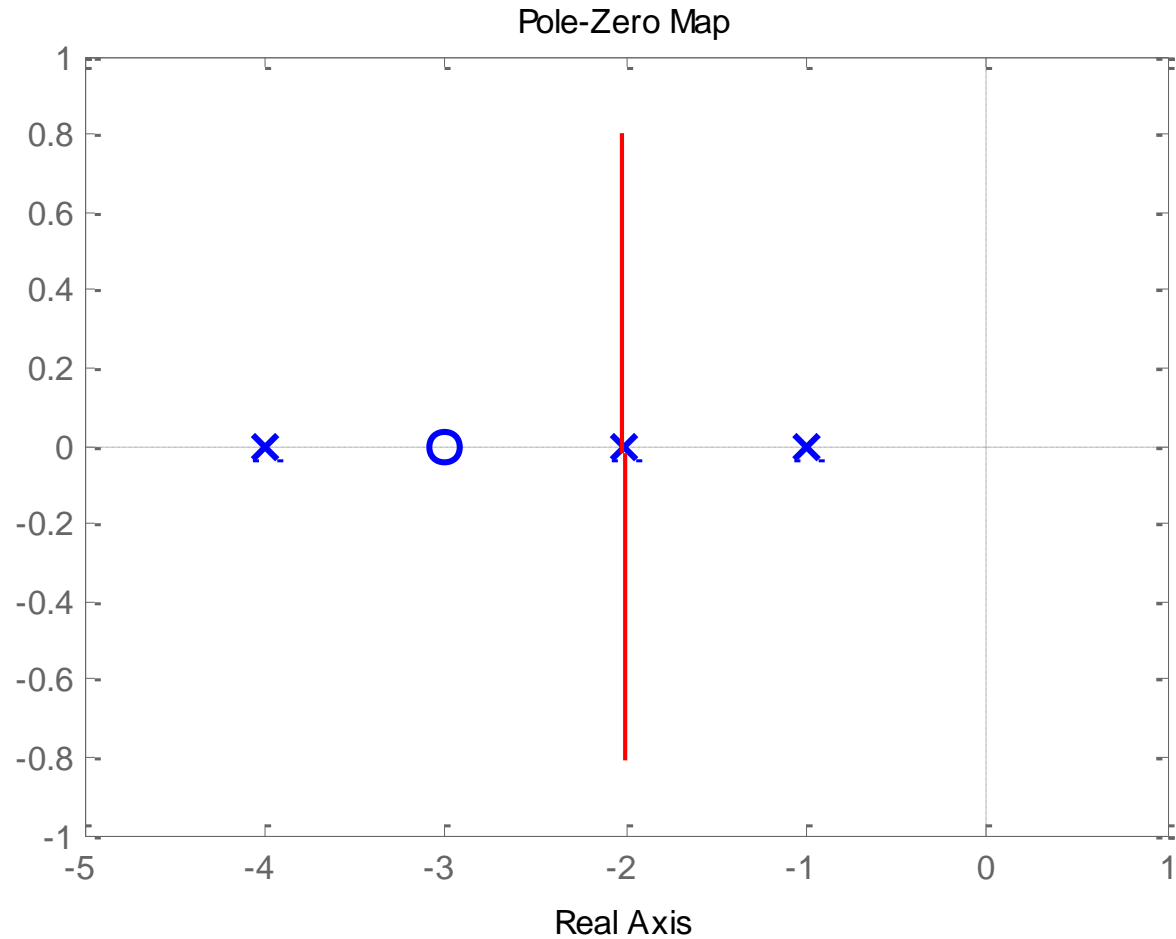


Example#2

- Step-3: Asymptotes

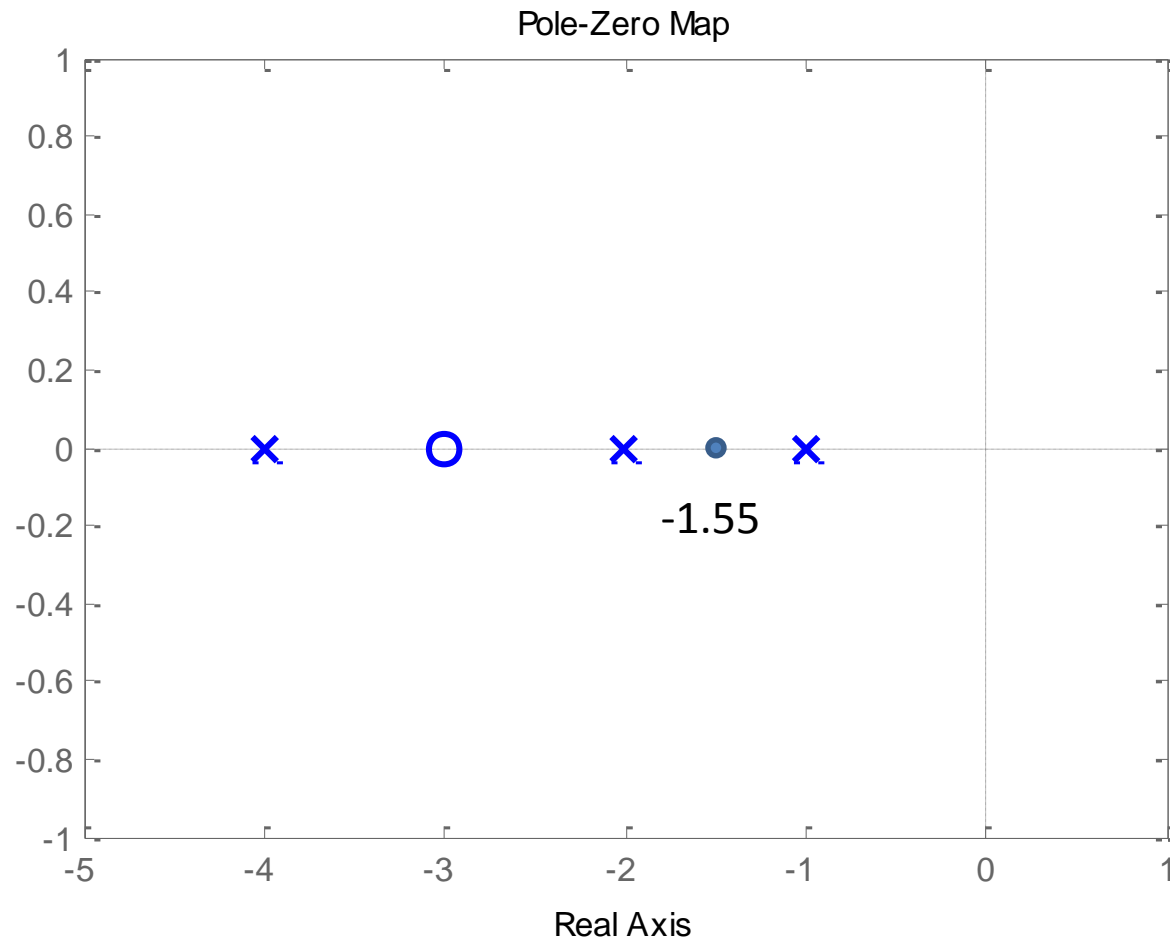
$$\psi = \pm 90^\circ$$

$$\sigma = -2$$

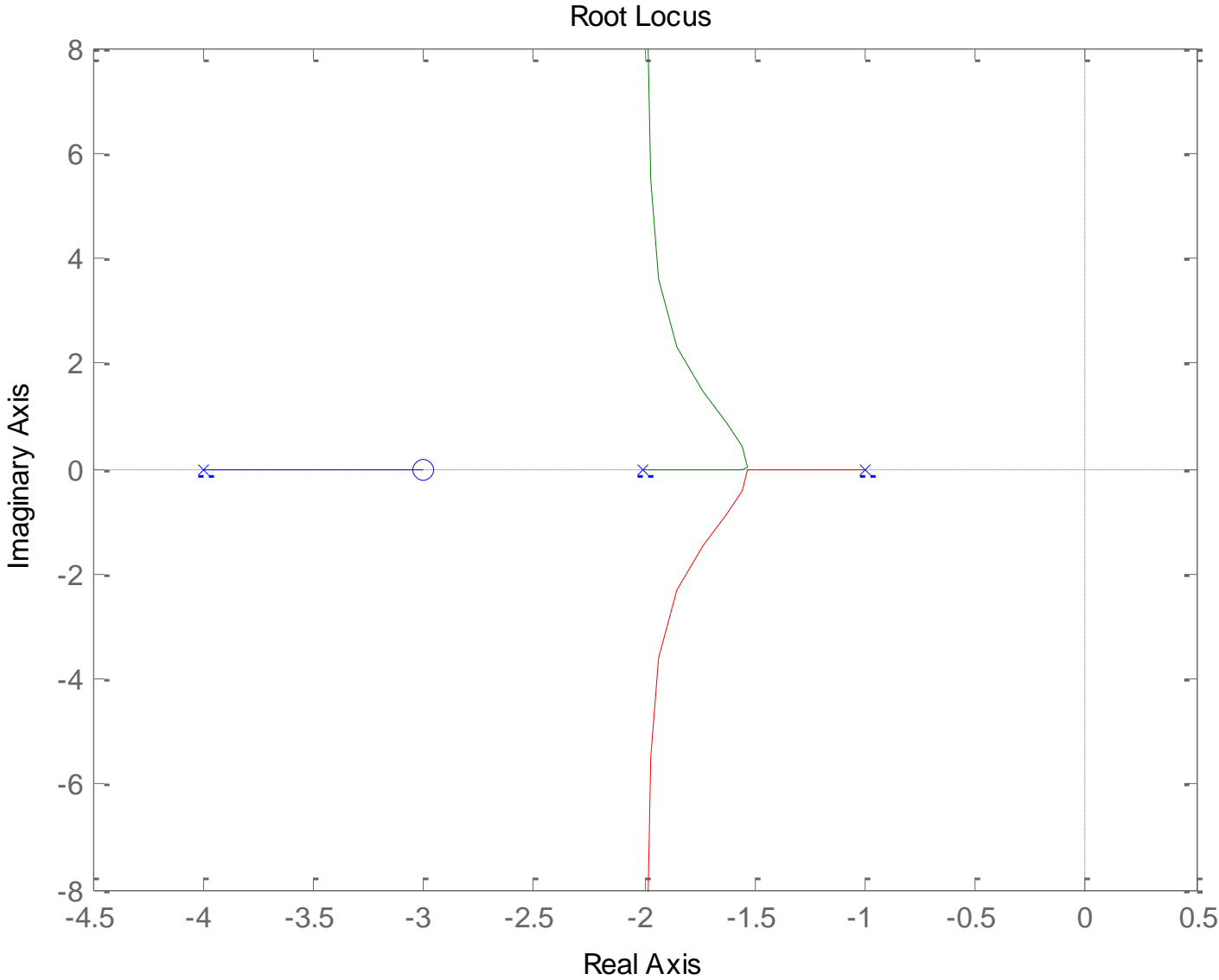


Example#2

- Step-4: breakaway point

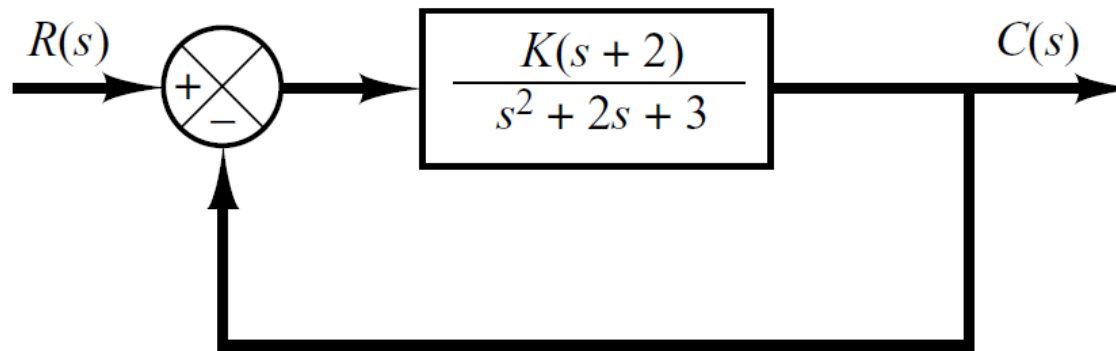


Example#2



Example

- Sketch the root-locus plot of following system with complex-conjugate open loop poles.



$$G(s) = \frac{K(s+2)}{s^2+2s+3}, \quad H(s) = 1$$

$G(s)$ has a pair of complex-conjugate poles at

$$s = -1 + j\sqrt{2}, \quad s = -1 - j\sqrt{2}$$

Example

- Step-1: Pole-Zero Map
- Step-2: Determine the root loci on real axis
- Step-3: Asymptotes

Example

- Step-4: Determine the angle of departure from the complex-conjugate open-loop poles.
 - The presence of a pair of complex-conjugate open-loop poles requires the determination of the angle of departure from these poles.
 - Knowledge of this angle is important, since the root locus near a complex pole yields information as to whether the locus originating from the complex pole migrates toward the real axis or extends toward the asymptote.

Example

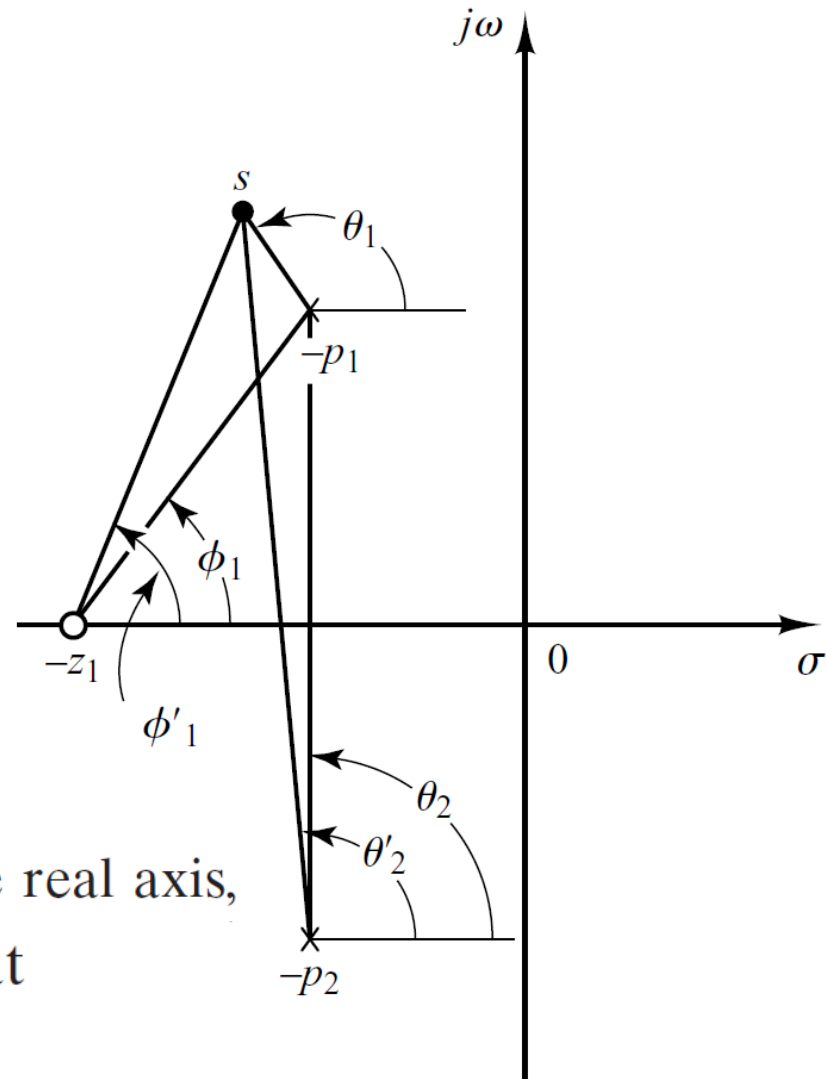
- Step-4:** Determine the angle of departure from the complex-conjugate open-loop poles.

The angle of departure is then

$$\begin{aligned}\theta_1 &= 180^\circ - \theta_2 + \phi_1 \\ &= 180^\circ - 90^\circ + 55^\circ = 145^\circ\end{aligned}$$

Since the root locus is symmetric about the real axis, the angle of departure from the pole at

$$s = -p_2 \text{ is } -145^\circ$$



Example

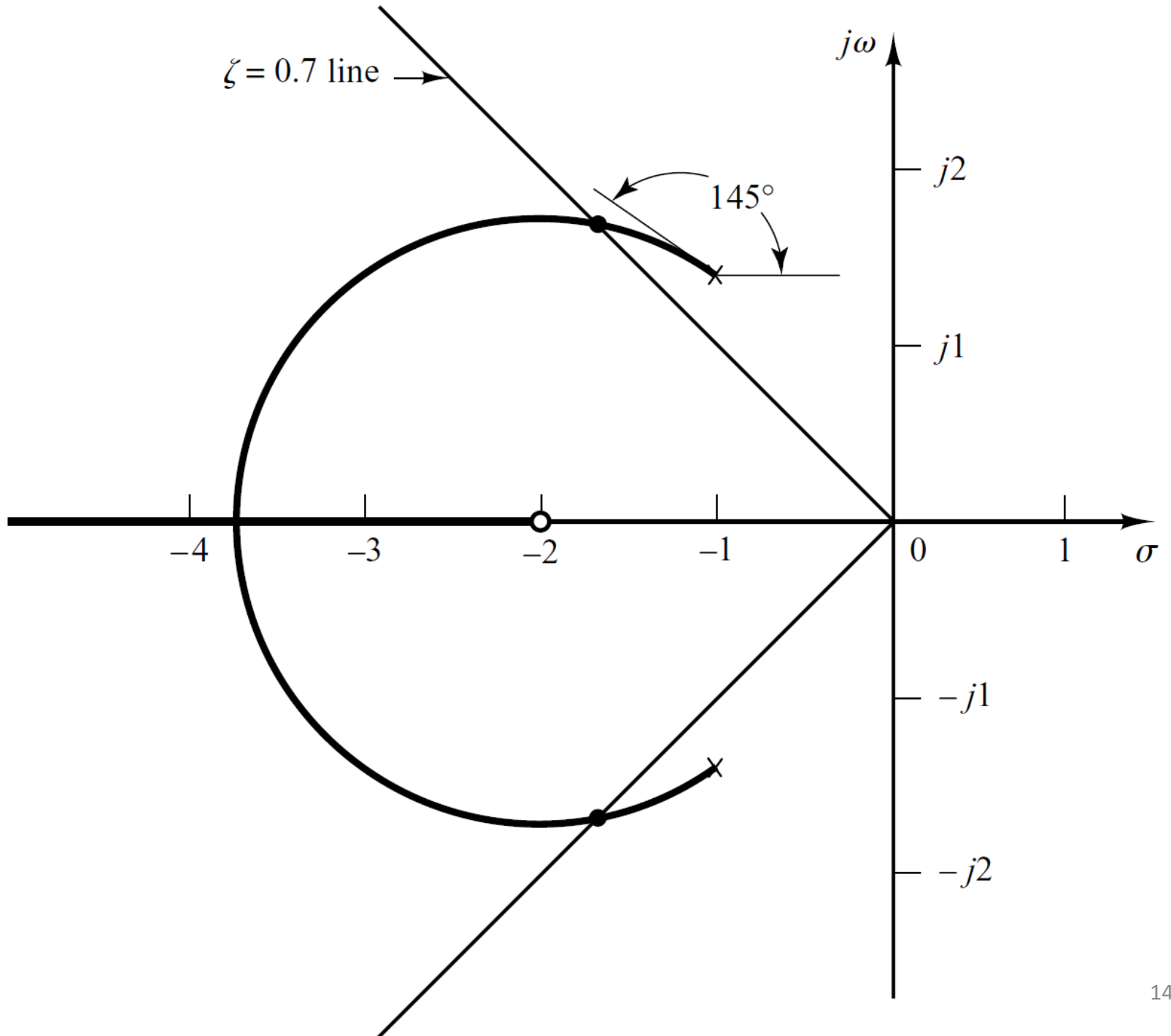
- Step-5: Break-in point

$$K = -\frac{s^2 + 2s + 3}{s + 2}$$

$$\frac{dK}{ds} = -\frac{(2s + 2)(s + 2) - (s^2 + 2s + 3)}{(s + 2)^2} = 0$$

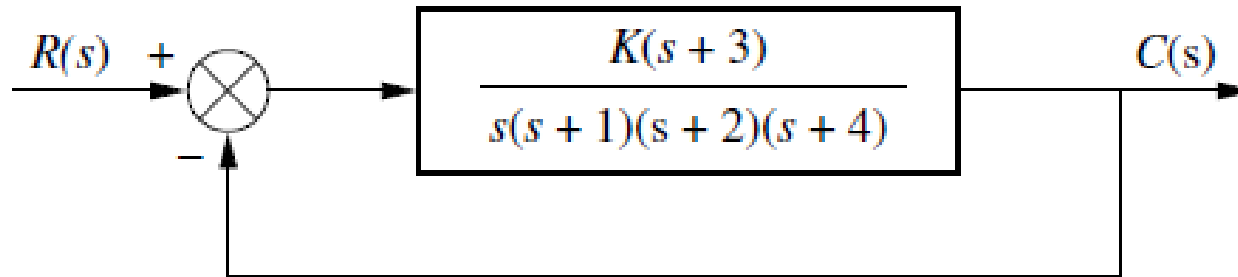
$$s^2 + 4s + 1 = 0$$

$$s = -3.7320 \quad \text{or} \quad s = -0.2680$$

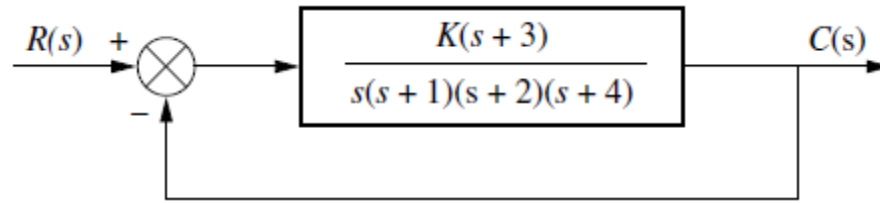


Root Locus of Higher Order System

- Sketch the Root Loci of following unity feedback system



$$G(s)H(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$



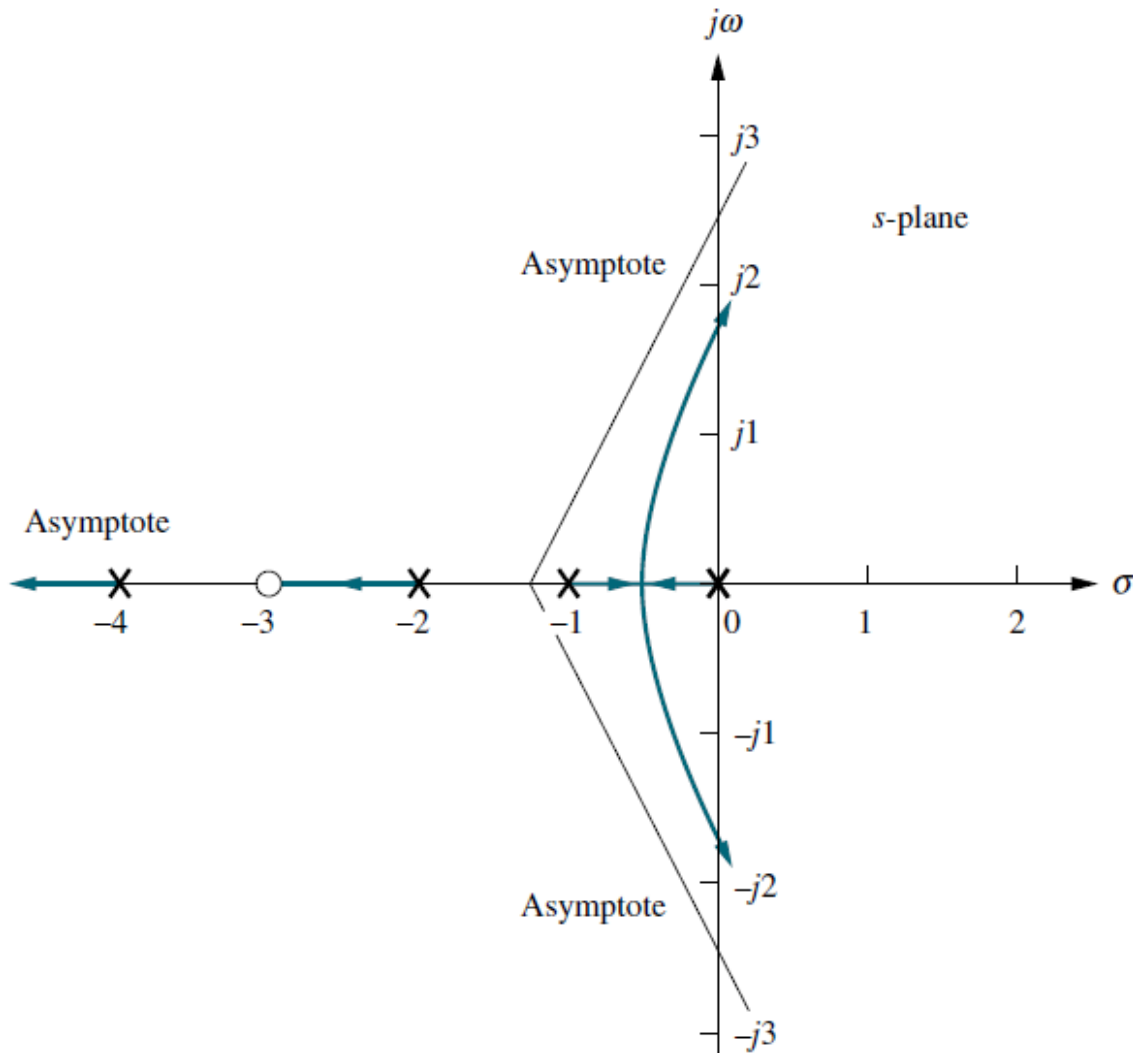
- Let us begin by calculating the asymptotes. The real-axis intercept is evaluated as;

$$\sigma_a = \frac{(-1 - 2 - 4) - (-3)}{4 - 1} = -\frac{4}{3}$$

- The angles of the lines that intersect at $-4/3$, given by

$$\begin{aligned} \theta_a &= \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \\ &= \pi/3 \quad \text{for } k = 0 \\ &= \pi \quad \text{for } k = 1 \\ &= 5\pi/3 \quad \text{for } k = 2 \end{aligned}$$

- The Figure shows the complete root locus as well as the asymptotes that were just calculated.



Example

$$1 + GH(s) = 1 + \frac{K(s + 1)}{s(s + 2)(s + 4)^2}$$

- a) Number of finite poles = $n = 4$.
- b) Number of finite zeros = $m = 1$.
- c) Number of asymptotes = $n - m = 3$.
- d) Number of branches or loci equals to the number of finite poles (n) = 4.
- e) The portion of the real-axis between, 0 and -2, and between, -4 and $-\infty$, lie on the root locus for $K > 0$.

- Using Eq. (v), the real-axis asymptotes intercept is evaluated as;

$$\sigma_a = \frac{(-2) + 2(-4) - (-1)}{n - m} = \frac{-10 + 1}{4 - 1} = -3$$

- The angles of the asymptotes that intersect at - 3, given by Eq. (vi), are;

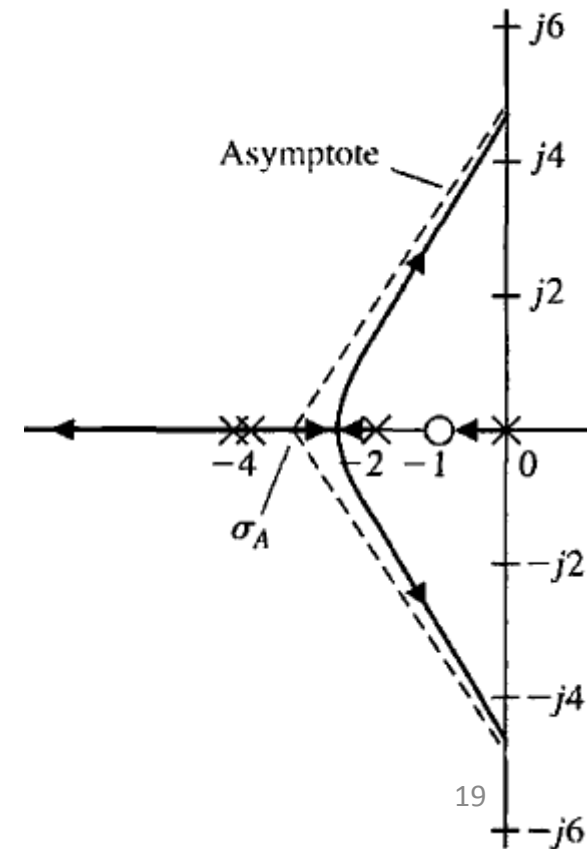
$$\theta_a = \frac{(2k + 1)\pi}{n - m} = \frac{(2k + 1)\pi}{4 - 1}$$

$$\text{For } K = 0, \quad \theta_a = 60^\circ$$

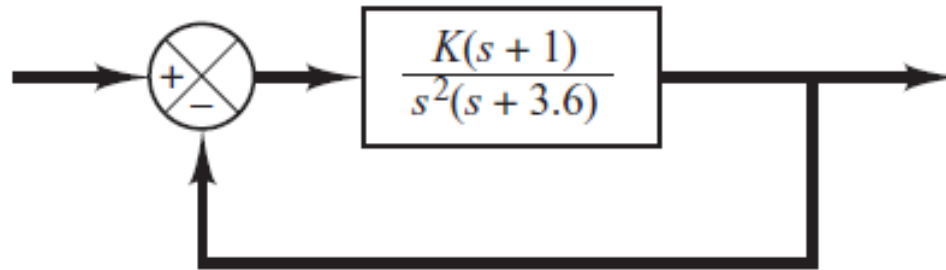
$$\text{For } K = 1, \quad \theta_a = 180^\circ$$

$$\text{For } K = 2, \quad \theta_a = 300^\circ$$

- The root-locus plot of the system is shown in the figure below.
- It is noted that there are three asymptotes. Since $n - m = 3$.
- The root loci must begin at the poles; two loci (or branches) must leave the double pole at $s = -4$.
- Using Eq. (vii), the breakaway point, σ , can be determine as;
- The solution of the above equation is $\sigma = -2.59$.



Example



- A root locus exists on the real axis between points $s = -1$ and $s = -3.6$.
- The intersection of the asymptotes and the real axis is determined as,

$$\sigma_a = \frac{0 + 0 + 3.6 - 1}{3 - 1} = \frac{2.6}{2} = -1.3$$

- The angles of the asymptotes that intersect at -1.3 , given by Eq. (vi), are;

$$\theta_a = \frac{(2k + 1)\pi}{n - m} = \frac{(2k + 1)\pi}{3 - 1}$$

For $K = 0$, $\theta_a = 90^\circ$
For $K = 1$, $\theta_a = -90^\circ$ or 270°

- Since the characteristic equation is $s^3 + 3.6s^2 + K(s + 1) = 0$

- We have $K = -\frac{s^3 + 3.6s^2}{s + 1} \longrightarrow$ (a)

$$K = -\frac{s^3 + 3.6s^2}{s + 1}$$

- The breakaway and break-in points:

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s + 1) - (s^3 + 3.6s^2)}{(s + 1)^2} = 0$$

$$\text{or} \quad s^3 + 3.3s^2 + 3.6s = 0$$

From which we get,

$$s = 0, \quad s = -1.65 + j0.9367, \quad s = -1.65 - j0.9367$$

- Point $s = 0$ corresponds to the actual breakaway point. But points $s = 1.65 \pm j0.9367$ neither breakaway nor break-in points, because the corresponding gain values K become complex quantities.

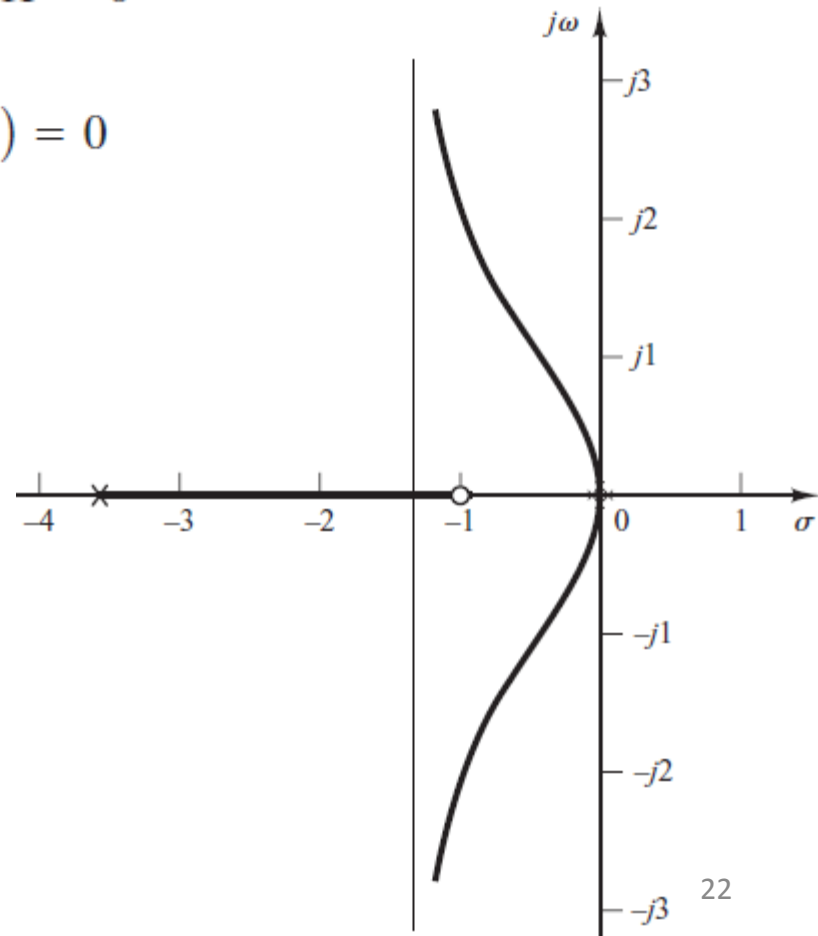
- To check the points where root-locus branches may cross the imaginary axis, substitute $s = j\omega$ into the characteristic equation, yielding.

$$(j\omega)^3 + 3.6(j\omega)^2 + Kj\omega + K = 0$$

or

$$(K - 3.6\omega^2) + j\omega(K - \omega^2) = 0$$

- Notice that this equation can be satisfied only if $\omega = 0, K = 0$.
- Because of the presence of a double pole at the origin, the root locus is tangent to the $j\omega$ axis at $k = 0$.
- The root-locus branches do not cross the $j\omega$ axis.
- The root loci of this system is shown in the Figure.



End of Lec

Note

- Determine the Breakaway and break in points

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

Solution

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2 - 8s + 15)}{(s^2 + 3s + 2)}$$

$$\frac{K(s^2 - 8s + 15)}{s^2 + 3s + 2} = -1$$

$$K = -\frac{(s^2 + 3s + 2)}{(s^2 - 8s + 15)}$$

- Differentiating K with respect to s and setting the derivative equal to zero yields;

$$\frac{dK}{ds} = -\frac{[(s^2 - 8s + 15)(2s + 3) - (s^2 + 3s + 2)(2s - 8)]}{(s^2 - 8s + 15)^2} = 0$$

$$11s^2 - 26s - 61 = 0$$

Hence, solving for s , we find the break-away and break-in points;

$$s = -1.45 \text{ and } 3.82$$