MODERN CONTROL SYSTEMS

Lecture 11

Pole Placement

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Pole Placement

- In this lecture we will discuss a design method commonly called the *pole-placement* or *pole-assignment technique.*

- We assume that all state variables are measurable and are available for feedback.

- If the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

Pole Placement

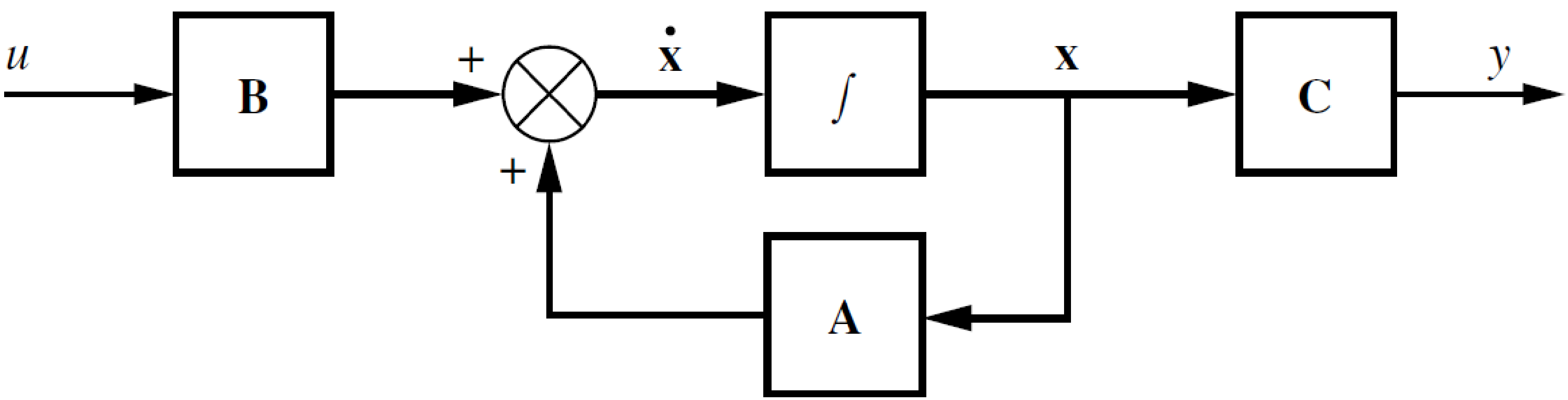
- The present design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements, such as speed, damping ratio, or bandwidth, as well as steady-state requirements.

- By choosing an appropriate gain matrix for state feedback, it is possible to force the system to have closed-loop poles at the desired locations, provided that the original system is completely state controllable.

Topology of Pole Placement

- Consider a plant represented in state space by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x}$$



Topology of Pole Placement

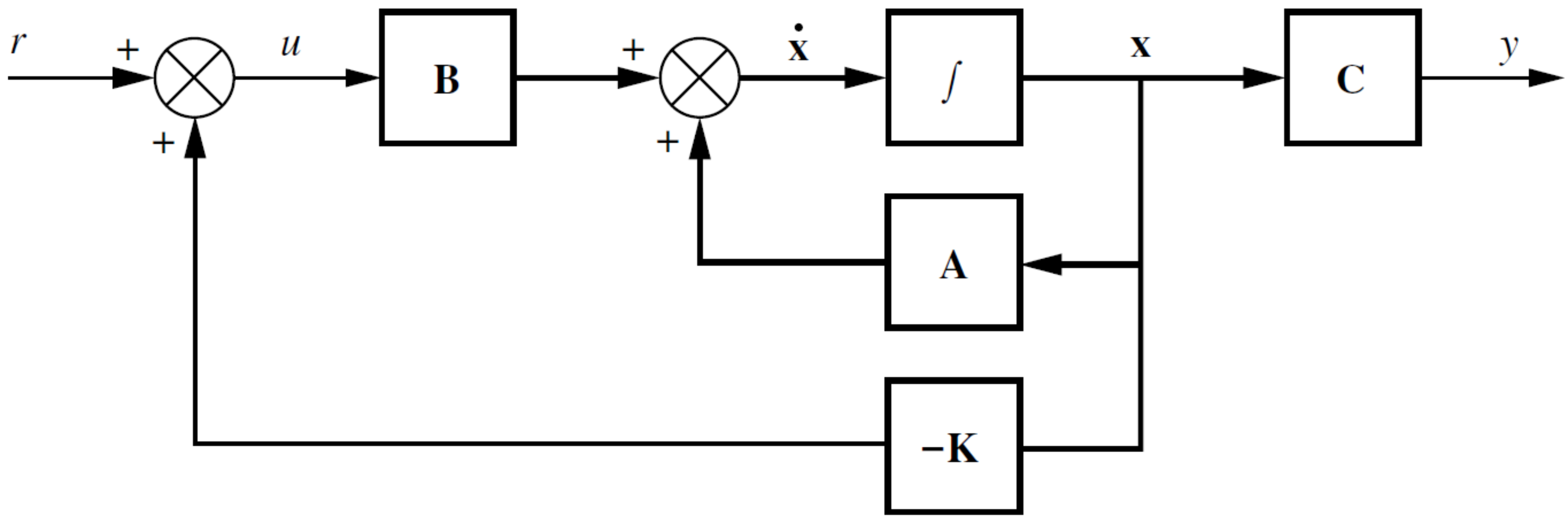
- In a typical feedback control system, the output, y, is fed back to the summing junction.

- It is now that the topology of the design changes. Instead of feeding back y, we feed back all of the state variables.

- If each state variable is fed back to the control, u, through a gain, k_i, there would be n gains, k_i, that could be adjusted to yield the required closed-loop pole values.

Topology of Pole Placement

- The feedback through the gains, k_i, is represented in following figure by the feedback vector **K**.



$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(r - \boldsymbol{K}\boldsymbol{x})$$

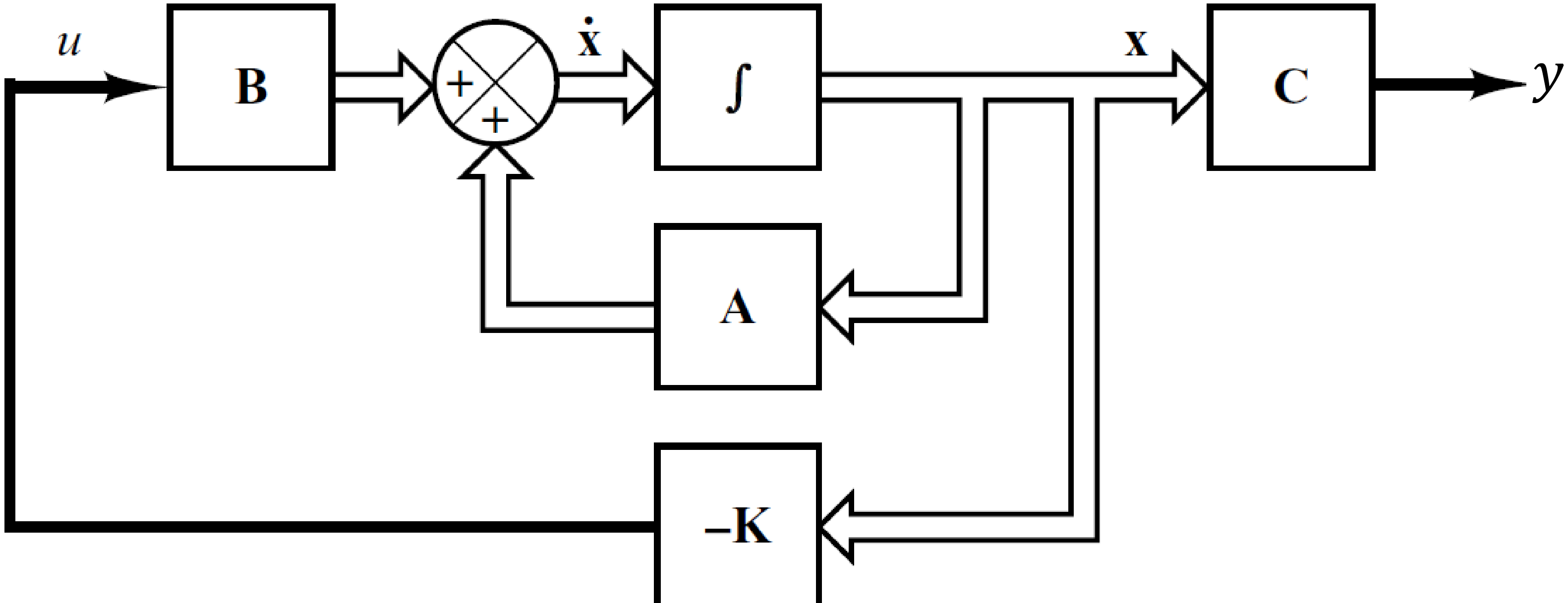
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}r - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}$$

$$y = \boldsymbol{C}\boldsymbol{x}$$

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}r$$

Pole Placement

- We will limit our discussions to single-input, single-output systems (i.e. we will assume that the control signal u(t) and output signal y(t) to be scalars).
- We will also assume that the reference input r(t) is zero.



$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} + \boldsymbol{B}r$$

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}$$

$$u = -\boldsymbol{Kx}$$

Pole Placement

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}$$

- The stability and transient response characteristics are determined by the eigenvalues of matrix **A-BK**.

- If matrix **K** is chosen properly Eigenvalues of the system can be placed at desired location.

- And the problem of placing the regulator poles (closed-loop poles) at the desired location is called a pole-placement problem.

Pole Placement

- There are three approaches that can be used to determine the gain matrix **K** to place the poles at desired location.

  - Direct Substitution Method.
  - Ackermann's formula.
  - Using Transformation Matrix P.

- All those method yields the same result.

Direct Substitution Method

Pole Placement (Direct Substitution Method)

- Following are the steps to be followed in this particular method.

1. Check the state controllability of the system

$$CM = [B \quad AB \quad A^2B \ \dots \quad A^{n-1}B]$$

Pole Placement (Direct Substitution Method)

- **Steps:**

1. Check the state controllability of the system.

$$C_T = [B \quad AB \quad A^2B \ \cdots \quad A^{n-1}B]$$

2. Define the state feedback gain matrix as

$$\boldsymbol{K} = [k_1 \quad k_2 \quad k_3 \cdots \quad k_n]$$

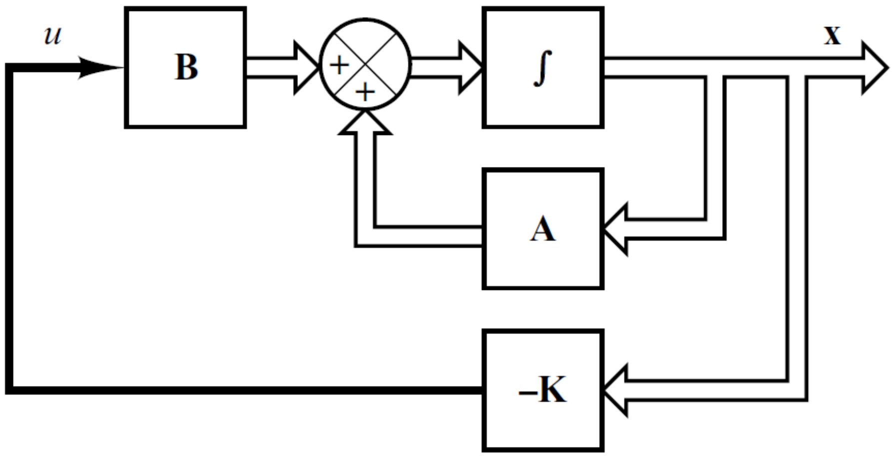
– And equating $|sI - A + BK|$ with desired characteristic equation.

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n$$

Example

- Consider the regulator system shown in following figure. The plant is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$



- The system uses the state feedback control u=-**K**x. The desired eigenvalues are $\mu_1 = -2 + j4$, $\mu_2 = -2 - j4$,$\mu_3 = -1$. Determine the state feedback gain matrix **K**.

Example

- Step-1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

- First, we need to check the controllability matrix of the system. Since the controllability matrix C_T is given by

$$C_T = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

- We find that rank(C_T)=3. Thus, the system is completely state controllable and arbitrary pole placement is possible.

- **Step-2:**
- Let **K** be

$$\boldsymbol{K} = [k_1 \quad k_2 \quad k_3]$$

$$|sI - A + BK| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \right|$$

$$= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

- Desired characteristic polynomial is obtained as

$$(s + 2 - 4j)(s + 2 + 4j)\ (s + 10) = s^3 + 14s^2 + 60s + 200$$

- Comparing the coefficients of powers of s

$$14 = (6 + k_3) \qquad k_3 = 8$$

$$60 = (5 + k_2) \qquad k_2 = 55$$

$$200 = 1 + k_1 \qquad k_1 = 199$$

Ackermann’s Formula

Pole Placement (Ackermann's Formula)

- Following are the steps to be followed in this particular method.

1. Check the state controllability of the system

$$CM = [B \quad AB \quad A^2B \; \ldots \quad A^{n-1}B]$$

Pole Placement (Ackermann's Formula)

- Following are the steps to be followed in this particular method.

2. Use Ackermann's formula to calculate **K**

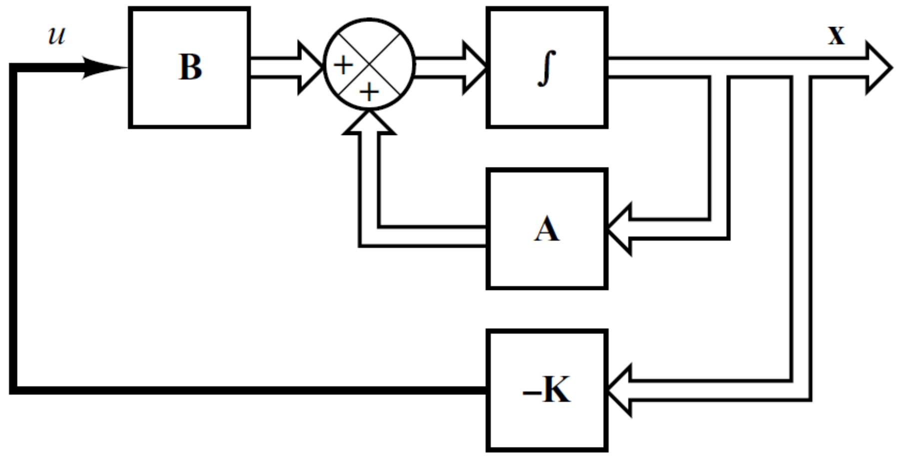
$$K = [0 \quad 0 \quad \cdots 0 \quad 1][B \quad AB \quad A^2B \ \cdots \quad A^{n-1}B]^{-1}\emptyset(A)$$

$$\emptyset(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1}A + \alpha_n I$$

Pole Placement (Ackermann's Formula)

- Example-1: Consider the regulator system shown in following figure. The plant is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$



- The system uses the state feedback control **u=-Kx**. The desired eigenvalues are $\mu_1 = -2 + j4$, $\mu_2 = -2 - j4$,$\mu_3 = -1$. Determine the state feedback gain matrix **K**.

Pole Placement (Using Transformation Matrix P)

- Example-1: Step-1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

- First, we need to check the controllability matrix of the system. Since the controllability matrix **CM** is given by

$$CM = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

- We find that rank(**CM**)=3. Thus, the system is completely state controllable and arbitrary pole placement is possible.

Pole Placement (Ackermann's Formula)

- Following are the steps to be followed in this particular method.

2. Use Ackermann's formula to calculate **K**

$$K = [0 \quad 0 \quad 1][B \quad AB \quad A^2B]^{-1}\emptyset(A)$$

$$\emptyset(A) = A^3 + \alpha_1 A^2 + \alpha_2 A + \alpha_3 I$$

- α_i are the coefficients of the desired characteristic polynomial.

$$(s + 2 - 4j)(s + 2 + 4j)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$\alpha_1 = 14, \qquad \alpha_2 = 60, \qquad \alpha_3 = 200$$

Pole Placement (Ackermann's Formula)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$\emptyset(A) = A^3 + 14A^2 + 60A + 200I$$

$$\emptyset(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\emptyset(A) = \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -34 & 117 \end{bmatrix}$$

Pole Placement (Ackermann's Formula)

$$[B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

$$\emptyset(A) = \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -34 & 117 \end{bmatrix}$$

$$K = [0 \quad 0 \quad 1][B \quad AB \quad A^2]^{-1}\emptyset(A)$$

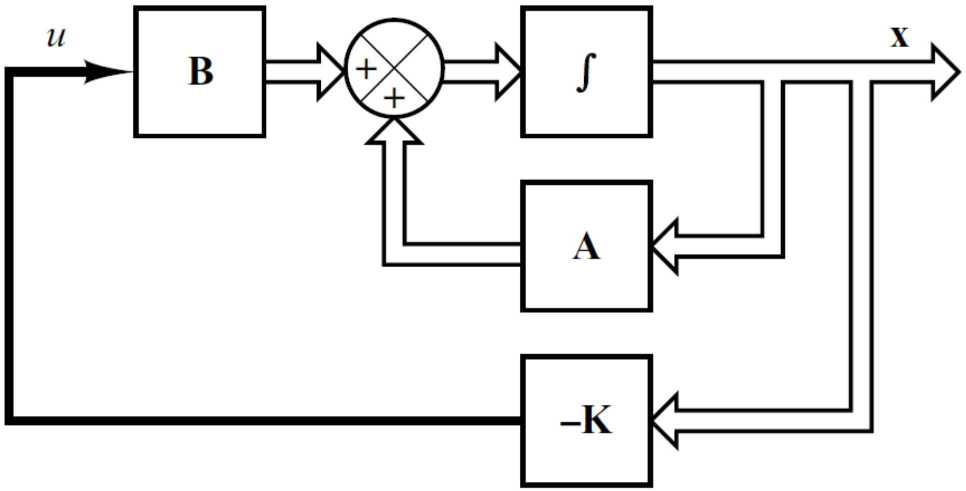
$$K = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -34 & 117 \end{bmatrix}$$

$$K = [199 \quad 55 \quad 8]$$

Pole Placement

- Example-2: Consider the regulator system shown in following figure. The plant is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$



- Determine the state feedback gain for each state variable to place the poles at -1+j, -1-j,-3. (Apply all methods)

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