# Robotics

Lecture 9

Velocity Kinematics


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# Velocity Kinematics

# Velocity Kinematics

| | Forward Kinematics | Inverse Kinematics |
|---|---|---|
| Given | • The time rate of joint variables | • The velocity of end-effector |
| Find | • The Cartesian velocity of end-effector in the global coordinate frame | • The time rate of joint variables |

# Forward Velocity Kinematics

# Forward Velocity Kinematics

- The forward velocity kinematics of a robot solves the problem of relating joint speeds $\dot{q}$ (of an *n DOF* robot is an *n × 1* vector)

$$\dot{q}_{n\times 1} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

to the end-effector speeds $\dot{X}$ (generally 6 × 1 vector).

$$\dot{\mathbf{X}} = [\, \overbrace{\dot{X}_n \quad \dot{Y}_n \quad \dot{Z}_n}^{\dot{d}_\mathrm{n}} \quad \overbrace{\omega_{Xn} \quad \omega_{Yn} \quad \omega_{Zn}}^{\omega_\mathrm{n}} \,]^T$$

$$= \begin{bmatrix} {}^0\dot{\mathbf{d}}_n \\ {}_0\omega_n \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{v}_n \\ {}_0\omega_n \end{bmatrix}$$

Linear Velocity

Angular Velocity

# Forward Velocity Kinematics

- The elements of end-effector speed vector $\dot{X}$ are linearly proportional to the elements of joint speed vector $\dot{q}$,

$$\dot{X} = J_{6\times n}\dot{q}_{n\times 1}$$

- *J(q) :* Jacobian matrix

# Jacobian

- Suppose a position and orientation vector of a manipulator is a function of 6 joint variables: (from forward kinematics)

$$X = h(q)$$

$$X = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix} = h\left(\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}\right)_{6\times 1} = \begin{bmatrix} h_1(q_1, q_2, \cdots, q_6) \\ h_2(q_1, q_2, \cdots, q_6) \\ h_3(q_1, q_2, \cdots, q_6) \\ h_4(q_1, q_2, \cdots, q_6) \\ h_5(q_1, q_2, \cdots, q_6) \\ h_6(q_1, q_2, \cdots, q_6) \end{bmatrix}_{6\times 1}$$

# Jacobian Matrix

Forward kinematics $$X_{6\times 1} = h(q_{n\times 1})$$

$$\dot{X}_{6\times 1} = \frac{d}{dt} h(q_{n\times 1}) = \frac{dh(q)}{dq}\frac{dq}{dt} = \frac{dh(q)}{dq}\dot{q}$$

Let $J = \dfrac{dh(q)}{dq}$

$$\dot{X}_{6\times 1} = J_{6\times n}\dot{q}_{n\times 1}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \left[\frac{dh(q)}{dq}\right]_{6\times n} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

# Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dfrac{dh(q)}{dq} \end{bmatrix}_{6\times n} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

$$J = \left(\frac{dh(q)}{dq}\right)_{6\times n} = \begin{bmatrix} \dfrac{\partial h_1}{\partial q_1} & \dfrac{\partial h_1}{\partial q_2} & \cdots & \dfrac{\partial h_1}{\partial q_n} \\ \dfrac{\partial h_2}{\partial q_1} & \dfrac{\partial h_2}{\partial q_2} & \cdots & \dfrac{\partial h_2}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial h_6}{\partial q_1} & \dfrac{\partial h_6}{\partial q_2} & \cdots & \dfrac{\partial h_6}{\partial q_n} \end{bmatrix}_{6\times n}$$

# Jacobian Matrix

The Jacobian Equation

$$\dot{X} = J_{6\times n}\dot{q}_{n\times 1}$$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} V \\ \omega \end{bmatrix}$$

$$V = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \qquad \omega = \begin{bmatrix} \omega_x = \dot{\phi} \\ \omega_y = \dot{\theta} \\ \omega_z = \dot{\psi} \end{bmatrix} \qquad \dot{q}_{n\times 1} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

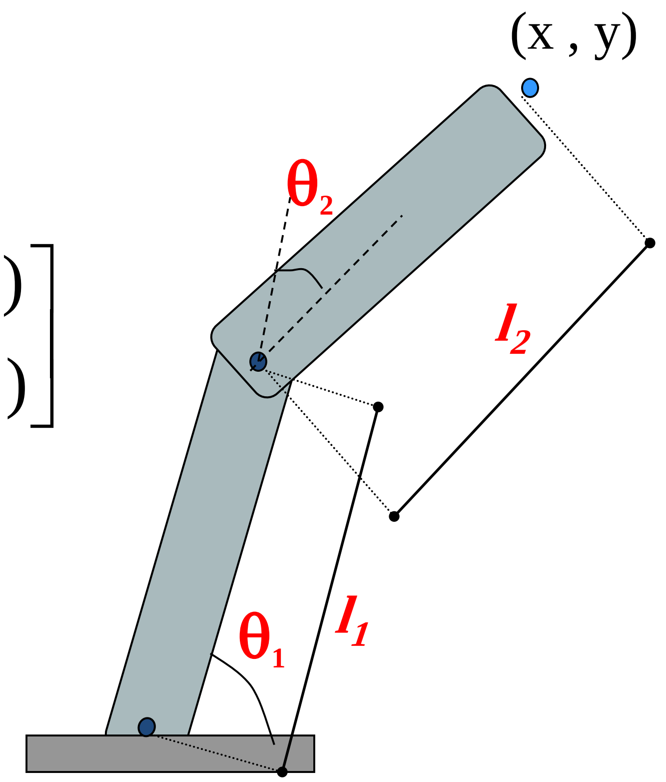
Linear velocity

Angular velocity

# Example

# Example

- 2-DOF planar robot arm
  - Given $l_1, l_2$, *Find: Jacobian*




$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \end{bmatrix} = \begin{bmatrix} h_1(\theta_1,\theta_2) \\ h_2(\theta_1,\theta_2) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} \dfrac{\partial h_1}{\partial \theta_1} & \dfrac{\partial h_1}{\partial \theta_2} \\ \dfrac{\partial h_2}{\partial \theta_1} & \dfrac{\partial h_2}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}$$

# Remember DH parmeter

$$A_i = \begin{bmatrix} \mathrm{c}\theta_\mathrm{i} & \text{-}\mathrm{c}\alpha_\mathrm{i}\mathrm{s}\theta_\mathrm{i} & \mathrm{s}\alpha_\mathrm{i}\mathrm{s}\theta_\mathrm{i} & \mathrm{a}_\mathrm{i}\mathrm{c}\theta_\mathrm{i} \\ \mathrm{s}\theta_\mathrm{i} & \mathrm{c}\theta_\mathrm{i}\mathrm{c}\alpha_\mathrm{i} & \text{-}\mathrm{s}\alpha_\mathrm{i}\mathrm{c}\theta_\mathrm{i} & \mathrm{a}_\mathrm{i}\mathrm{s}\theta_\mathrm{i} \\ 0 & \mathrm{s}\alpha_\mathrm{i} & \mathrm{c}\alpha_\mathrm{i} & \mathrm{d}_\mathrm{i} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The transformation matrix T

$$T_0^i = A_1 A_2 \ldots\ldots A_i$$

# Jacobian Matrix

$$J = \begin{bmatrix} J_1 & J_2 & \cdots & J_n \end{bmatrix}$$

where if joint *(i)* is revolute

$$J_i = \begin{bmatrix} Z_{i-1} \times (O_n - O_{i-1}) \\ Z_{i-1} \end{bmatrix}$$

and if joint *(i)* is prismatic

$$J_i = \begin{bmatrix} Z_{i-1} \\ 0 \end{bmatrix}$$

Where $Z_i$ is the first three elements in the 3rd column of the $T_0^i$ matrix, and $O_i$ is the first three elements in the 4th column of the $T_0^i$ matrix.

# Example

2-DOF planar robot arm. Given *l1, l2 , Find: Jacobian*

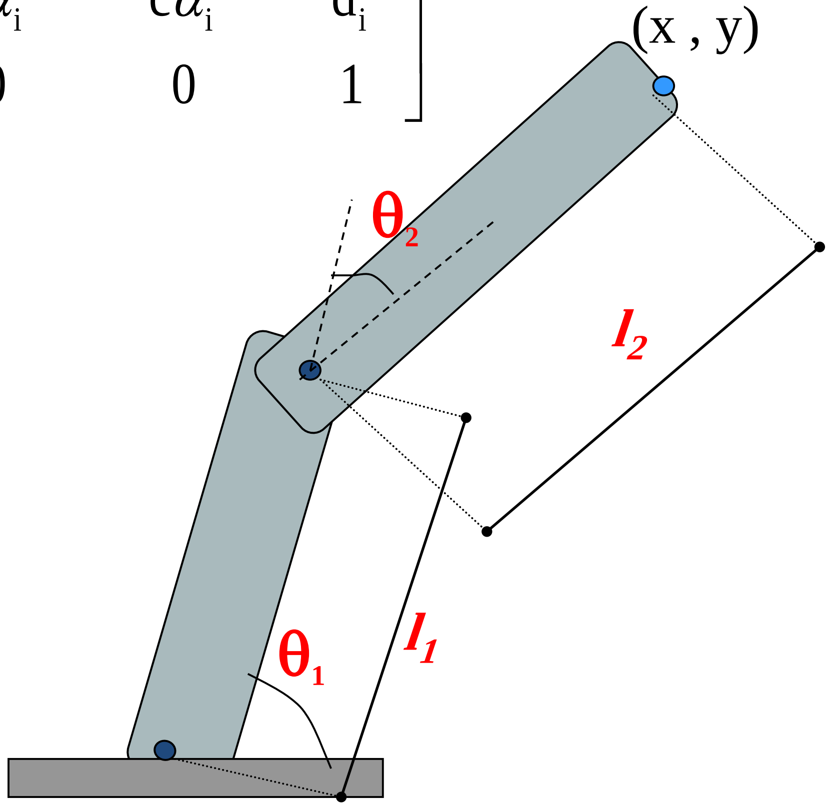
| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1^*$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2^*$ |

$$A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* variable

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$T_1^0 = A_1.$$

$$T_2^0 = A_1 A_2 = \begin{array}{c} \begin{array}{cccc} \phantom{c_{12}} & \phantom{-s_{12}} & Z_2 & O_2 \end{array} \\ \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Where $(\theta_1 + \theta_2)$ denoted by $\theta_{12}$ and $\cos(\theta_1 + \theta_2)$ by $c_{12}$

$$Z_0 = Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, O_1 = \begin{bmatrix} a_1 \cos\theta_1 \\ a_1 \sin\theta_1 \\ 0 \end{bmatrix}, O_2 = \begin{bmatrix} a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ z_0 \end{bmatrix}, J_2 = \begin{bmatrix} z_1 \times (o_2 - o_1) \\ z_1 \end{bmatrix}$$

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$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ z_0 \end{bmatrix}$$

$$Z_0 \times (o_2 - o_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

# Note: Cross Products

- If $A = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, B = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$

- Then the cross product

$$A \times B = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ -(a_x b_z - a_z b_x) \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$J_2 = \begin{bmatrix} Z_1 \times (o_2 - o_1) \\ Z_1 \end{bmatrix}$$

$$Z_1 \times (o_2 - o_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 \cos(\theta_1 + \theta_2) \\ a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ a_2 \cos(\theta_1 + \theta_2) & a_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad J_2 = \begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The required Jacobian matrix $\boldsymbol{J}$

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 \end{bmatrix}$$

# Example

# Stanford Manipulator

The DH parameters are:

| Link | $d_i$ | $a_i$ | $\alpha_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $-90$ | $\star$ |
| 2 | $d_2$ | 0 | $+90$ | $\star$ |
| 3 | $\star$ | 0 | 0 | 0 |
| 4 | 0 | 0 | $-90$ | $\star$ |
| 5 | 0 | 0 | $+90$ | $\star$ |
| 6 | $d_6$ | 0 | 0 | $\star$ |

* joint variable

$$A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$Z_1$ $O_1$

$$T_0^1 = A_1$$

$$T_0^2 = A_1 A_2 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & -d_2s_1 \\ s_1c_2 & c_1 & s_1s_2 & d_2c_1 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_0^3 = A_1A_2A_3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2 - d_2s_1 \\ s_1c_2 & c_1 & s_1s_2 & d_3s_1s_2 + d_2c_1 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$Z_2$ $O_2$ $Z_3$ $O_3$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} z_2 = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix} z_3 = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix} \quad O_0 = O_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} O_2 = \begin{bmatrix} -d_2s_1 \\ d_2c_1 \\ 0 \end{bmatrix} O_3 = \begin{bmatrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \end{bmatrix}$$

$$T_0^4 = A_1 A_2 A_3 A_4 \qquad T_0^5 = A_1 A_2 A_3 A_4 A_5 \qquad T_0^6 = A_1 A_2 A_3 A_4 A_5 A_6$$

$$T_0^4 = \begin{bmatrix} \text{c1c2c4}-\text{s1s4} & -\text{c1s2} & -\text{c1c2s4}-\text{s1}*\text{c4} & \text{c1s2d3-s1d2} \\ \text{s1c2c4+c1s4} & -\text{s1s2} & -\text{s1c2s4+c1c4} & \text{s1s2d3+c1*d2} \\ -\text{s2c4} & -\text{c2} & \text{s2s4} & \text{c2*d3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \qquad O_4 = \begin{bmatrix} d_3 c_1 s_2 - d_2 s_1 \\ d_3 s_1 s_2 + d_2 c_1 \\ d_3 c_2 \end{bmatrix}$$

$$T_0^5 = \left[\begin{array}{ccc} \text{(c1c2c4-s1s4)c5-c1s2s5} & \text{c1c2s4+s1c4} & \text{(c1c2c4-s1s4)s5+c1s2c5} \\ \text{(s1c2c4+c1s4)c5-s1s2s5} & \text{s1c2s4-c1c4} & \text{(s1c2c4+c1s4)s5+s1s2c5} \\ \text{-s2c4c5-c2s5} & \text{-s2s4} & \text{-s2c4s5+c2c5} \\ 0 & 0 & 0 \end{array}\right.$$

$$z_5 = \begin{bmatrix} c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5 \\ s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 \\ -s_2c_4s_5 + c_2c_5 \end{bmatrix}$$

$$z_5 = \begin{bmatrix} c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5 \\ s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 \\ -s_2c_4s_5 + c_2c_5 \end{bmatrix}$$

$$T_0^5 = \begin{bmatrix} \text{(c1c2c4-s1s4)c5-c1s2s5} & \text{c1c2s4+s1c4} & \text{(c1c2c4-s1s4)s5+c1s2c5} & \text{c1s2d3-s1d2} \\ \text{(s1c2c4+c1s4)c5-s1s2s5} & \text{s1c2s4-c1c4} & \text{(s1c2c4+c1s4)s5+s1s2c5} & \text{s1s2d3+c1d2} \\ \text{-s2c4c5-c2s5} & \text{-s2s4} & \text{-s2c4s5+c2c5} & \text{c2d3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$z_5 = \begin{bmatrix} c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5 \\ s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 \\ -s_2c_4s_5 + c_2c_5 \end{bmatrix}$$

$$O_5 = \begin{bmatrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \end{bmatrix}$$

$$T_0^6 = \begin{bmatrix} (\text{c6c5c1c2c4-c6c5s1s4-c6c1s2s5+s6c1c2s4+s6s1c4} & \text{-c5c1c2c4+s6c5s1s4+s6c1s2s5+c6c1c2s4+c6s1c4} & \text{s5c1c2c4-s5s1s4+c1s2c5} & \text{d6s5c1c2c4-d6s5s1s4+d6c1s2c5+c1s2d3-s1d2} \\ \text{c6c5s1c2c4+c6c5c1s4-c6s1s2s5+s6s1c2s4-s6c1c4} & \text{s6c5s1c2c4-s6c5c1s4+s6s1s2s5+c6s1c2s4-c6c1c4} & \text{s5s1c2c4+s5c1s4+s1s2c5} & \text{d6s5s1c2c4+d6s5c1s4+d6s1s2c5+s1s2d3+c1d2} \\ \text{-c6s2c4c5-c6c2s5-s2s4s6} & \text{s6s2c4c5+s6c2s5-s2s4c6} & \text{-s2c4s5+c2c5} & \text{-d6s2c4s5+d6c2c5+c2d3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T6 = [  c6c5c1c2c4-c6c5s1s4-c6c1s2s5+s6c1c2s4+s6s1c4, -c5c1c2c4+s6c5s1s4+s6c1s2s5+c6c1c2s4+c6s1c4, s5c1c2c4-s5s1s4+c1s2c5, d6s5c1c2c4-d6s5s1s4+d6c1s2c5+c1s2d3-s1d2]

[  c6c5s1c2c4+c6c5c1s4-c6s1s2s5+s6s1c2s4-s6c1c4, -s6c5s1c2c4-s6c5c1s4+s6s1s2s5+c6s1c2s4-c6c1c4, s5s1c2c4+s5c1s4+s1s2c5, d6s5s1c2c4+d6s5c1s4+d6s1s2c5+s1s2d3+c1d2]

[  -c6s2c4c5-c6c2s5-s2s4s6, s6s2c4c5+s6c2s5-s2s4c6, -s2c4s5+c2c5, -d6s2c4s5+d6c2c5+c2d3]

[    0, 0, 0, 1]

$$Z_6 = \begin{bmatrix} \text{s5c1c2c4}-\text{s5s1s4+c1s2c5} \\ \text{s5s1c2c4+s5c1s4+s1s2c5} \\ -\text{s2c4s5+c2c5} \end{bmatrix}$$

$$O_6 = \begin{bmatrix} \text{d6s5c1c2c4} - \text{d6s5s1s4} + \text{d6c1s2c5} + \text{c1s2d3} - \text{s1d2} \\ \text{d6s5s1c2c4} + \text{d6s5c1s4} + \text{d6s1s2c5} + \text{s1s2d3} + \text{c1d2} \\ -\text{d6s2c4s5} + \text{d6c2c5} + \text{c2d3} \end{bmatrix}$$

$$J_1 = \begin{bmatrix} z_0 \times (o_6 - o_0) \\ z_0 \end{bmatrix}, J_2 = \begin{bmatrix} z_1 \times (o_6 - o_1) \\ z_1 \end{bmatrix}$$

Joints 1,2 are revolute

$$J_3 = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}$$

Joint 3 is prismatic

$$J_4 = \begin{bmatrix} z_3 \times (o_6 - o_3) \\ z_3 \end{bmatrix}, J_5 = \begin{bmatrix} z_4 \times (o_6 - o_4) \\ z_4 \end{bmatrix}, J_6 = \begin{bmatrix} z_5 \times (o_6 - o_5) \\ z_5 \end{bmatrix}$$

The required Jacobian matrix $\boldsymbol{J}$

$$J = [J_1 \;\; J_2 \;\; J_3 \;\; J_4 \quad J_5 \quad J_6]$$

*End of Lec*

*End of Lec*