Robotics

Lecture 11

Path Planning

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Introduction

- Path planning includes three tasks:
  - Defining a geometric curve for the end-effector between two points.
  - Defining a rotational motion between two orientations.
  - Defining a time function for variation of a coordinate between two given values.

- All of these three definitions are called path planning.
a path of the tip point of a 2R manipulator between points P1 and P2 to avoid two obstacles.
Cubic Path

• A cubic function is the simplest polynomial to determine the time behavior of a variable between two given values, rest-to-rest.

• A cubic path in joint space for the joint variable $q(t)$, or in Cartesian space for a Cartesian coordinate $q(t)$, between two points $q(t_0)$ and $q(t_f)$ is

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
A cubic polynomial has four coefficients. Therefore, it can satisfy the position and velocity constraints at the initial and final points.

For simplicity, we call the value of the variable, the position, and the rate of the variable, the velocity.

Assume that the position and velocity of a variable at the initial time \( t_0 \) and at the final time \( t_f \) are given as:

\[
\begin{align*}
q(t_0) &= q_0 \\
\dot{q}(t_0) &= \dot{q}_0 \\
q(t_f) &= q_f \\
\dot{q}(t_f) &= \dot{q}_f
\end{align*}
\]
Substituting the boundary conditions in the position and velocity equations of the joint variable

\[ q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \]
\[ \dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2 \]

generates four equations for the coefficients of the path.

\[
\begin{bmatrix}
1 & t_0 & t_0^2 & t_0^3 \\
0 & 1 & 2t_0 & 3t_0^2 \\
1 & t_f & t_f^2 & t_f^3 \\
0 & 1 & 2t_f & 3t_f^2
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
=
\begin{bmatrix}
q_0 \\
q'_0 \\
q_f \\
q'_f
\end{bmatrix}
\]
Their solutions are

\[ a_0 = -\frac{q_1 t_0^2 (t_0 - 3t_f) + q_0 t_f^2 (3t_0 - t_f)}{(t_f - t_0)^3} - t_0 t_f \frac{q_0 t_f + q_1 t_0}{(t_f - t_0)^2} \]

\[ a_1 = 6t_0 t_f \frac{q_0 - q_1}{(t_f - t_0)^3} + \frac{q_0 t_f \left( t_f^2 + t_0 t_f - 2t_0^2 \right) + q_1 t_0 \left( 2t_f^2 - t_0^2 - t_0 t_f \right)}{(t_f - t_0)^3} \]

\[ a_2 = -\frac{q_0 (3t_0 + 3t_f) + q_1 (-3t_0 - 3t_f)}{(t_f - t_0)^3} - \frac{q_1 t_f \left( t_0 t_f - 2t_0^2 + t_f^2 \right) + q_0 \left( 2t_f^2 - t_0^2 - t_0 t_f \right)}{(t_f - t_0)^3} \]

\[ a_3 = \frac{2q_0 - 2q_1 + q_0' (t_f - t_0) + q_1' (t_f - t_0)}{(t_f - t_0)^3} \]
In case that \( t_0 = 0 \), the coefficients simplify to

\[
\begin{align*}
  a_0 &= q_0 \\
  a_1 &= q'_0 \\
  a_2 &= 3 (q_f - q_0) - \left(2q'_0 + q'_f\right) t_f \\
  a_3 &= -2 (q_f - q_0) + \left(q'_0 + q'_f\right) t_f \\
\end{align*}
\]
Example
Assume \( q(0) = 10 \text{ deg}, \ q(1) = 45 \text{ deg}, \) and \( \dot{q}(0) = \dot{q}(1) = 0. \) The coefficients of the cubic path are

\[
\begin{align*}
a_0 &= 10 & a_1 &= 0 & a_2 &= 105 & a_3 &= -70
\end{align*}
\]

that generate a path for the variable as

\[
q(t) = 10 + 105t^2 - 70t^3 \text{ deg}.
\]
Example
Assume the angle of a joint starts from $\theta(0) = 10 \text{ deg}$, $\dot{\theta}(0) = 12 \text{ deg} / \text{s}$ and ends at $\theta(2) = 45 \text{ deg}$, $\dot{\theta}(0) = 0$. The coefficients of a cubic path for this motion are:

$$a_0 = 10 \quad a_1 = 12 \quad a_2 = \frac{81}{2} \quad a_3 = \frac{-29}{2}$$

The kinematics of this path are

$$\theta(t) = 10 + 12t + 40.5t^2 - 14.5t^3 \text{ deg}$$
$$\dot{\theta}(t) = 81t - 43.5t^2 + 12 \text{ deg} / \text{s}$$
$$\ddot{\theta}(t) = 81 - 87t \text{ deg} / \text{s}^2$$
Manipulator Motion by Joint Path

Having the joint variables as functions of time, and employing the forward kinematics of manipulators, allows us to calculate the path of motion for the end-effector.
Example
A 2R robot moving along a line

Let us consider a 2R manipulator with $L_1 = L_2 = 0.25$ that its tip point is supposed to move on a given line $Y = f(X)$

$Y = -0.25998X + 0.3705$
Assume that the first angle is moving between 45 deg and 135 deg in 10 sec

\[ 45 \text{ deg} < \theta_1 < 135 \text{ deg} \]

based on a cubic path.

\[ \theta_1 = \frac{\pi}{4} + \frac{3\pi}{200} t^2 - \frac{\pi}{1000} t^3 \quad 0 < t < 10 \text{ sec} \]

The elbow joint R will move on a circle and at the beginning is at:

\[ X_{R_1} = 0.25 \cos \frac{\pi}{4} = 0.17678 \]

\[ Y_{R_1} = 0.25 \sin \frac{\pi}{4} = 0.17678 \]
Point $P_1$ must be on the line $(Y = -0.25998X + 0.3705)$ at a distance $d = 0.25$ from $R_1$.

\[
d = \sqrt{(X - 0.17678)^2 + (Y - 0.17678)^2}
\]

\[
= \sqrt{(X - 0.17678)^2 + (-0.25998X + 0.3705 - 0.17678)^2} = 0.25
\]

Therefore, $P_1$ is at:

\[
X_{P_1} = 0.41122 \quad Y_{P_1} = 0.26359
\]

and **initial values of angles** $\varphi$ and $\theta_2$ are:

\[
\varphi = \arctan \frac{Y_{P_1} - Y_{R_1}}{X_{P_1} - X_{R_1}} = \arctan \frac{0.26359 - 0.17678}{0.41122 - 0.17678}
\]

\[
= 0.35463 \text{ rad} \approx 20.319 \text{ deg}
\]

\[
\theta_2 = \theta_1 - \varphi = \frac{\pi}{4} - 0.35463
\]

\[
= 0.43077 \text{ rad} \approx 24.681 \text{ deg}
\]
The elbow joint $R$ at the final position is at:

\[
X_{R_2} = 0.25 \cos \frac{3\pi}{4} = -0.17678
\]

\[
Y_{R_2} = 0.25 \sin \frac{3\pi}{4} = 0.17678
\]

Point $P_2$ must be on the line $(Y = -0.25998X + 0.3705)$ at a distance $d = 0.25$ from $R_2$

\[
d = \sqrt{(X + 0.17678)^2 + (Y - 0.17678)^2}
\]

\[
= \sqrt{(X + 0.17678)^2 + (-0.25998X + 0.3705 - 0.17678)^2} = 0.25
\]

Therefore, $P_2$ is at:

\[
X_{P_2} = -2.8188 \times 10^{-2} \quad Y_{P_2} = 0.37783
\]
and final values of angles $\varphi$ and $\theta_2$ are:

$$
\varphi = \arctan \frac{Y_{P2} - Y_{R2}}{X_{P2} - X_{R2}} = \arctan \frac{0.37783 - 0.17678}{-2.8188 \times 10^{-2} + 0.17678} \\
= 0.93432 \text{ rad} \approx 53.533 \text{ deg}
$$

$$
\theta_2 = \theta_1 - \varphi = \frac{3\pi}{4} - 0.93432 \\
= 1.4219 \text{ rad} \approx 81.469 \text{ deg}
$$

To determine $\theta_2$ during the motion, we should follow the same procedure. Let us find the position of the elbow joint R as a function of $\theta_1$.

$$
X_R = 0.25 \cos \theta_1 \quad Y_R = 0.25 \sin \theta_1
$$

The tip point P must be on the line \((Y = -0.25998X + 0.3705)\) at a distance \(d\) from the elbow joint \(R\).

\[
d = \sqrt{(X_P - 0.25 \cos \theta_1)^2 + (Y_P - 0.25 \sin \theta_1)^2}
\]

\[
= \sqrt{(X_P - 0.25 \cos \theta_1)^2 + (-0.25998X_P + 0.3705 - 0.25 \sin \theta_1)^2}
\]

Solution of this equation for \(X_P\) and substitution in \((Y = -0.25998X + 0.3705)\) provides the coordinates \((X_P, Y_P)\) of the tip point P during the motion. Then, the angle \(\varphi\) and \(\theta_2\) would be:

\[
\varphi = \arctan \frac{Y_P - Y_R}{X_P - X_R} = \arctan \frac{Y_P - 0.25 \sin \theta_1}{X_P - 0.25 \cos \theta_1}
\]

\[
\theta_2 = \theta_1 - \varphi = \theta_1 - \arctan \frac{Y_P - 0.25 \sin \theta_1}{X_P - 0.25 \cos \theta_1}
\]
- Therefore, to make the point P moving along the line:

\[ Y = -0.25998X + 0.3705 \]

- While \( \theta_1 \) is varying as:

\[
\theta_1 = \frac{\pi}{4} + \frac{3\pi}{200} t^2 - \frac{\pi}{1000} t^3 \quad 0 < t < 10 \text{ sec}
\]

- The angle \( \theta_2 \) must vary according to:

\[
\theta_2 = \theta_1 - \varphi = \theta_1 - \arctan \left( \frac{Y_P - 0.25 \sin \theta_1}{X_P - 0.25 \cos \theta_1} \right)
\]
End of Lec