

# Robotics

## Lecture 6

### Inverse Kinematics

Emam Fathy

Department of Electrical and Control Engineering

email: [emfmz@aast.edu](mailto:emfmz@aast.edu)

[http://www.aast.edu/cv.php?disp\\_unit=346&ser=68525](http://www.aast.edu/cv.php?disp_unit=346&ser=68525)

# Inverse Kinematics (IK)

“Given a goal position find the joint angles for the robot arm”

# Inverse Kinematics

- The inverse kinematics is needed in the control of manipulators.
- Solving the inverse kinematics is computationally expensive and generally takes a very long time in the real time control of manipulators.
- IK generally harder than FK
- Sometimes no analytical solution
- Sometimes multiple solutions
- Sometimes no solution
  - Outside workspace

# Inverse kinematics

```
graph TD; IK[Inverse kinematics] --> AM[Analytical Method]; IK --> NM[Numerical Method]; AM --> GS[Geometric solution]; AM --> AS[Algebraic solution];
```

## Analytical Method

Joint variables solved according to given configuration data

## Numerical Method

Joint variables obtained by numerical techniques

## Geometric solution

For simple structures, 2-DOF

## Algebraic solution

For more links and in 3 dimensions

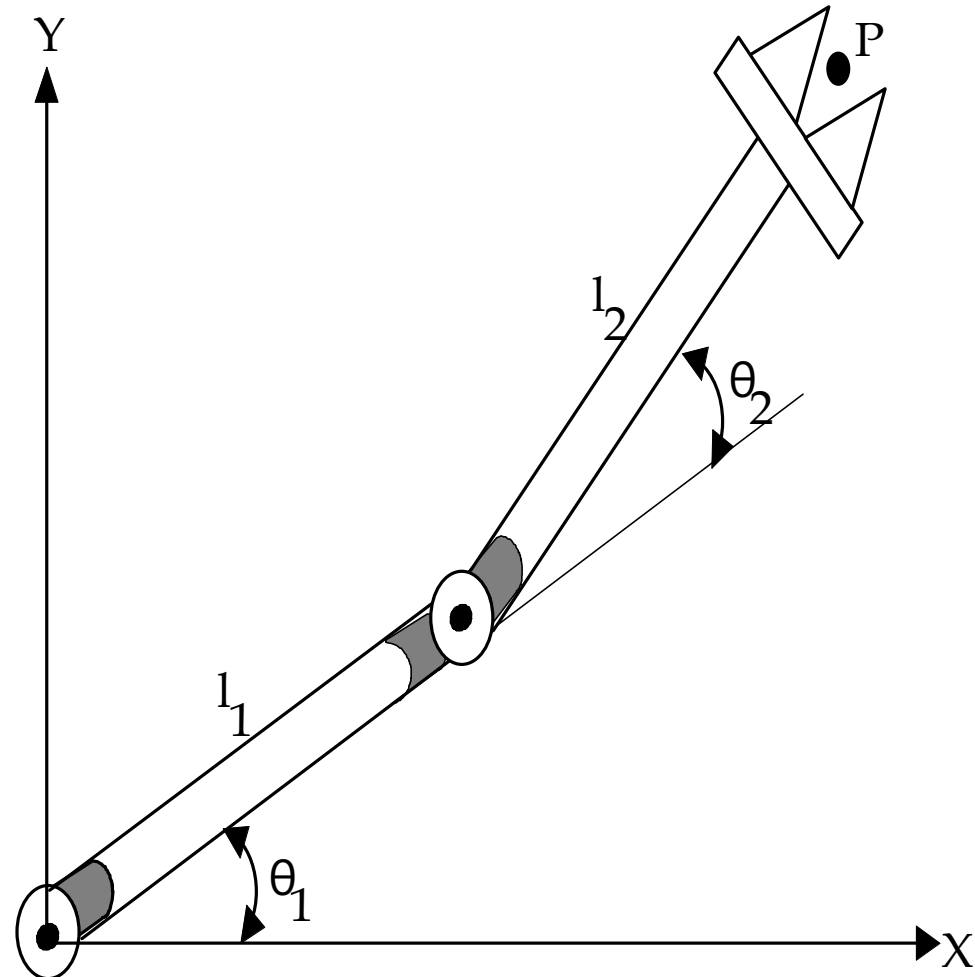
# Geometric Solution Approach

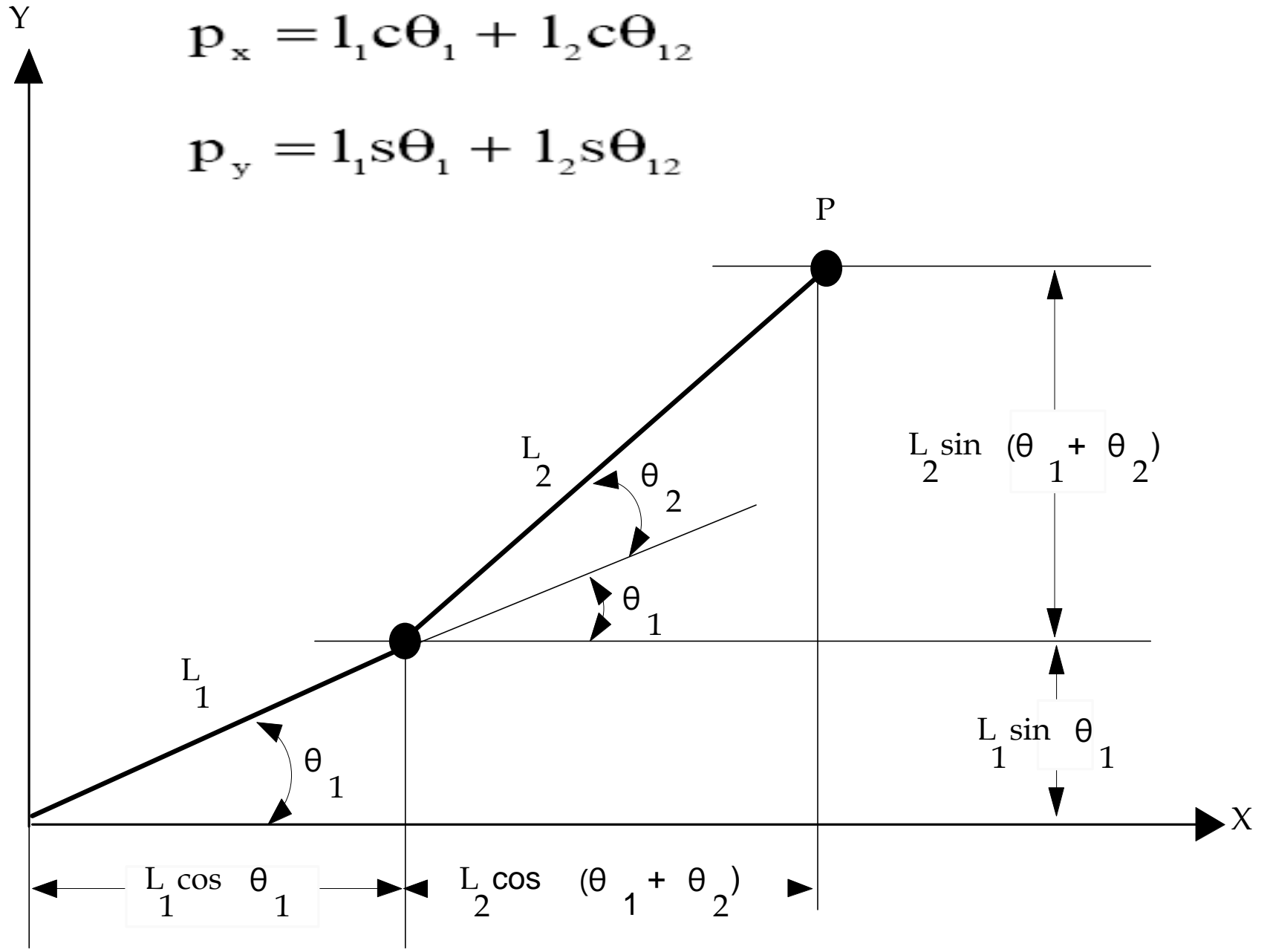
# Geometric Solution Approach

- It is applied to the simple robot structures, such as, 2-DOF planer manipulator whose joints are both revolute.
- In the shown Figure, the components of point P ( $p_x$ ,  $p_y$ ) are determined as follows.

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$





$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$

- The solution of  $\theta_2$ -can be computed from summation of squaring both previous equations

$$p_x^2 = l_1^2 c^2\theta_1 + l_2^2 c^2\theta_{12} + 2l_1 l_2 c\theta_1 c\theta_{12}$$

$$p_y^2 = l_1^2 s^2\theta_1 + l_2^2 s^2\theta_{12} + 2l_1 l_2 s\theta_1 s\theta_{12}$$

$$p_x^2 + p_y^2 = l_1^2 (c^2\theta_1 + s^2\theta_1) + l_2^2 (c^2\theta_{12} + s^2\theta_{12}) + 2l_1 l_2 (c\theta_1 c\theta_{12} + s\theta_1 s\theta_{12})$$

- Since  $c^2\theta_1 + s^2\theta_1 = 1$ , the equation is simplified as:

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1 l_2 (c\theta_1 [c\theta_1 c\theta_2 - s\theta_1 s\theta_2] + s\theta_1 [s\theta_1 c\theta_2 + c\theta_1 s\theta_2])$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1 l_2 (c^2\theta_1 c\theta_2 - c\theta_1 s\theta_1 s\theta_2 + s^2\theta_1 c\theta_2 + c\theta_1 s\theta_1 s\theta_2)$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1 l_2 (c\theta_2 [c^2\theta_1 + s^2\theta_1])$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c\theta_2$$



$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2c\theta_2$$

and so

$$c\theta_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

Since,  $c^2\theta_i + s^2\theta_i = 1$  ( $i = 1, 2, 3, \dots$ ),  $s\theta_2$  is obtained as

$$s\theta_2 = \pm \sqrt{1 - \left( \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2} \right)^2}$$

Finally, two possible solutions for  $\theta_2$  can be written as

$$\theta_2 = A \tan 2 \left( \pm \sqrt{1 - \left( \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2} \right)^2}, \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2} \right)$$

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12} \quad \dots\dots\dots 1$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12} \quad \dots\dots\dots 2$$

- The solution of  $\theta_1$  multiply each side of equation 1 by  $c\theta_1$  and equation 2 by  $s\theta_1$  and add the resulting equations in order to find the solution of  $\theta_1$  in terms of link parameters and the known variable  $\theta_2$ .

$$c\theta_1 p_x = l_1 c^2\theta_1 + l_2 c^2\theta_1 c\theta_2 - l_2 c\theta_1 s\theta_1 s\theta_2$$

$$s\theta_1 p_y = l_1 s^2\theta_1 + l_2 s^2\theta_1 c\theta_2 + l_2 s\theta_1 c\theta_1 s\theta_2$$

$$c\theta_1 p_x + s\theta_1 p_y = l_1 (c^2\theta_1 + s^2\theta_1) + l_2 c\theta_2 (c^2\theta_1 + s^2\theta_1)$$

The simplified equation obtained as follows.

$$c\theta_1 p_x + s\theta_1 p_y = l_1 + l_2 c\theta_2 \quad \dots\dots\dots 3$$

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12} \quad \dots\dots\dots 1$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12} \quad \dots\dots\dots 2$$

- Multiply each side of equation 1 by  $-s\theta_1$  and equation 2 by  $c\theta_1$  and add the resulting equations

$$-s\theta_1 p_x = -l_1 s\theta_1 c\theta_1 - l_2 s\theta_1 c\theta_1 c\theta_2 + l_2 s^2\theta_1 s\theta_2$$

$$c\theta_1 p_y = l_1 s\theta_1 c\theta_1 + l_2 c\theta_1 s\theta_1 c\theta_2 + l_2 c^2\theta_1 s\theta_2$$

$$-s\theta_1 p_x + c\theta_1 p_y = l_2 s\theta_2 (c^2\theta_1 + s^2\theta_1)$$

The simplified equation is given by

$$-s\theta_1 p_x + c\theta_1 p_y = l_2 s\theta_2 \quad \dots\dots\dots 4$$

Now, multiply each side of equation 3 by  $p_x$  and equation 4 by  $p_y$  and add the resulting equations in order to obtain  $c\theta_1$ .

$$c\theta_1 p_x^2 + s\theta_1 p_x p_y = p_x (l_1 + l_2 c\theta_2)$$

$$-s\theta_1 p_x p_y + c\theta_1 p_y^2 = p_y l_2 s\theta_2$$

$$c\theta_1 (p_x^2 + p_y^2) = p_x (l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2$$

and so

$$c\theta_1 = \frac{p_x (l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}$$

$$c\theta_1 = \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}$$

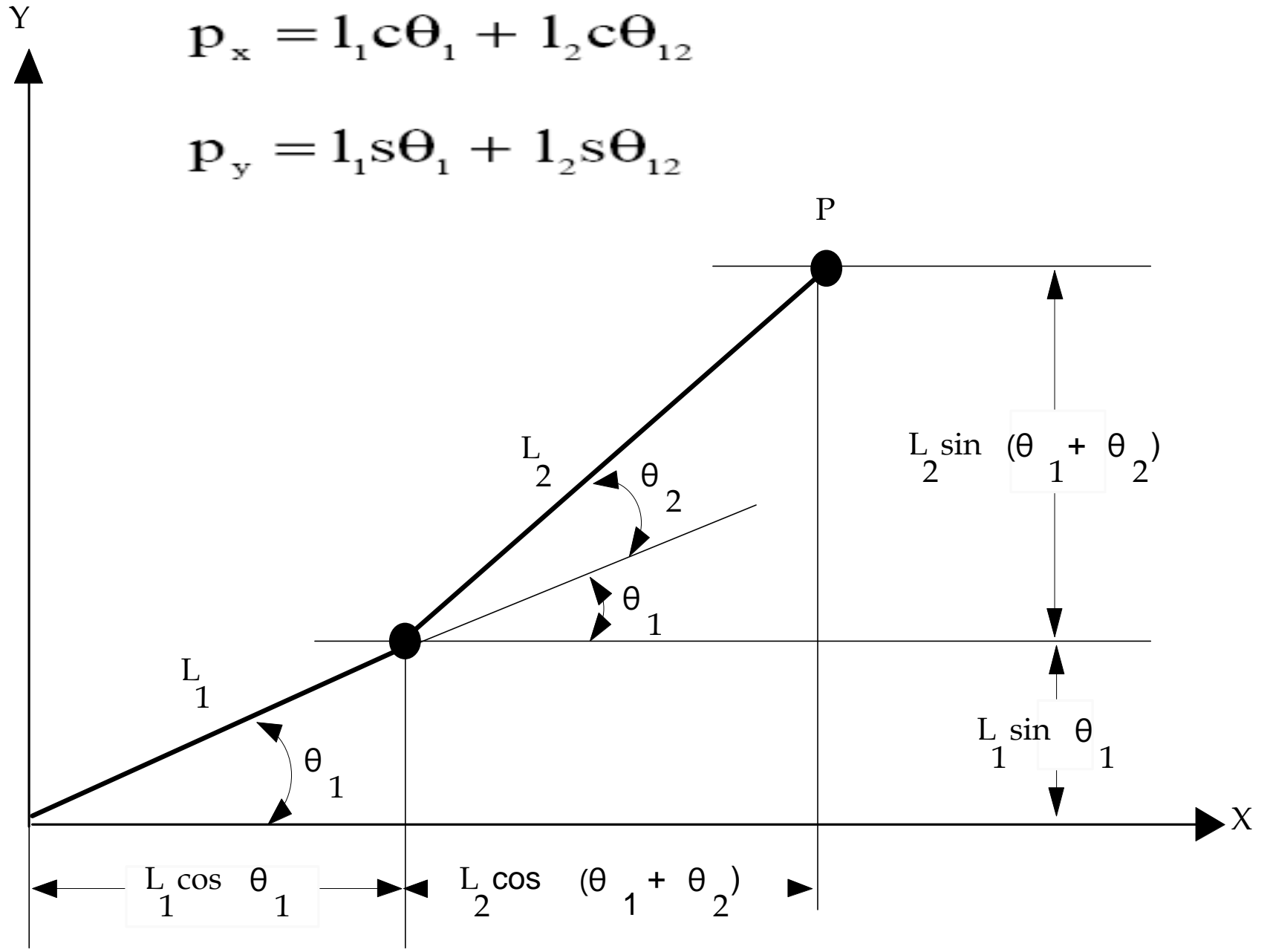
$s\theta_1$  is obtained as

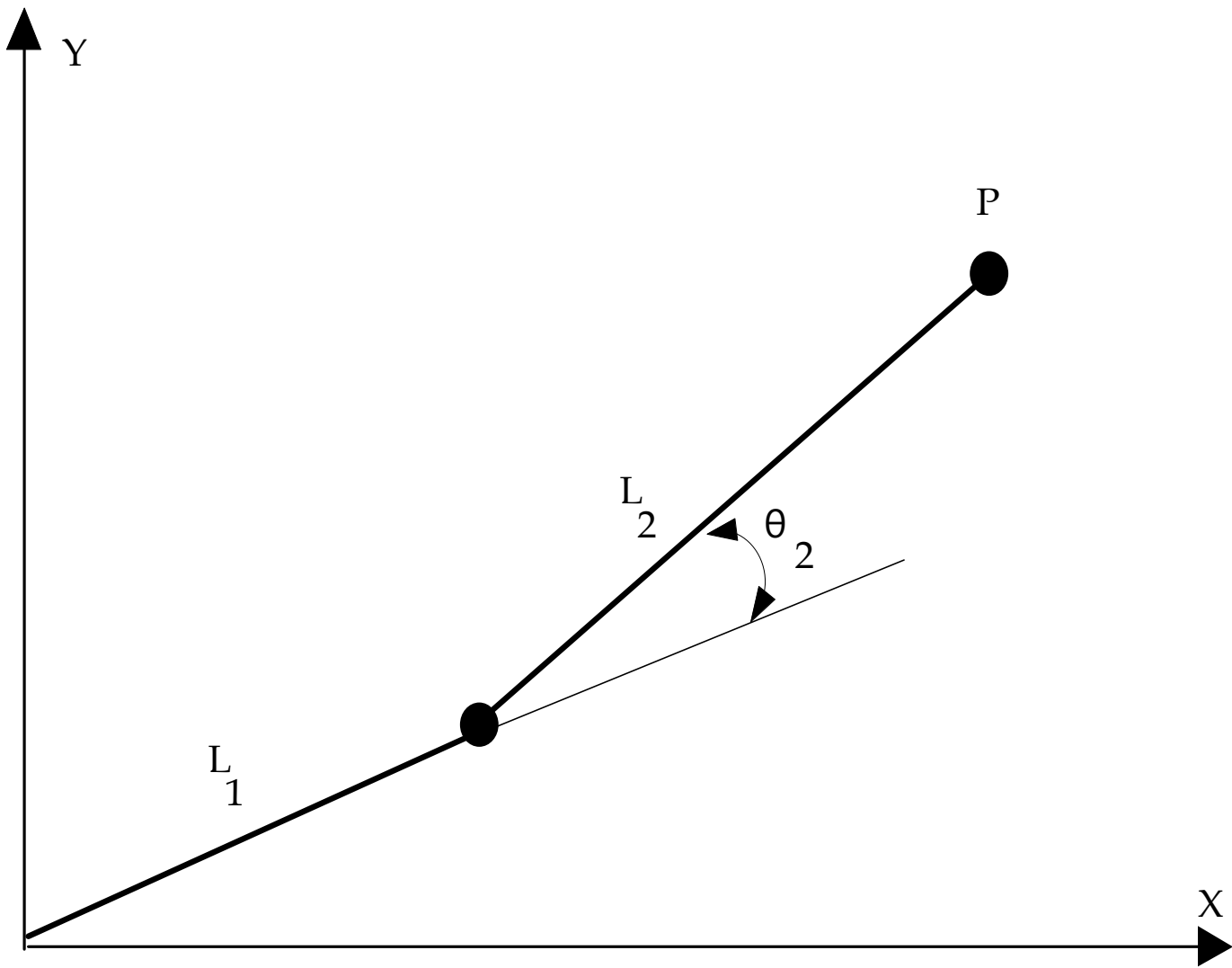
$$s\theta_1 = \pm \sqrt{1 - \left( \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2} \right)^2}$$

As a result, two possible solutions for  $\theta_1$  can be written

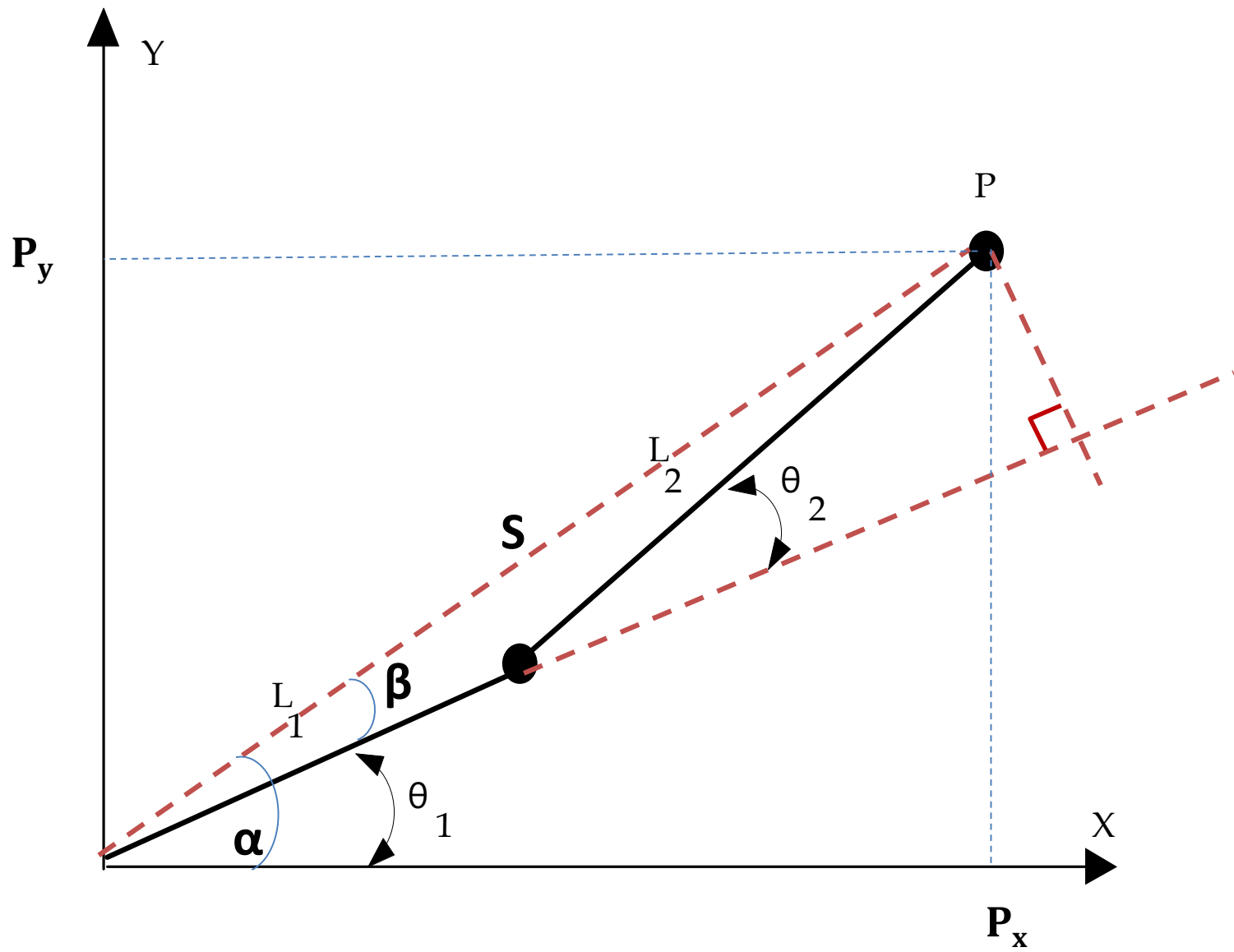
$$\theta_1 = A \tan 2 \left( \pm \sqrt{1 - \left( \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2} \right)^2}, \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2} \right)$$

# Another solution



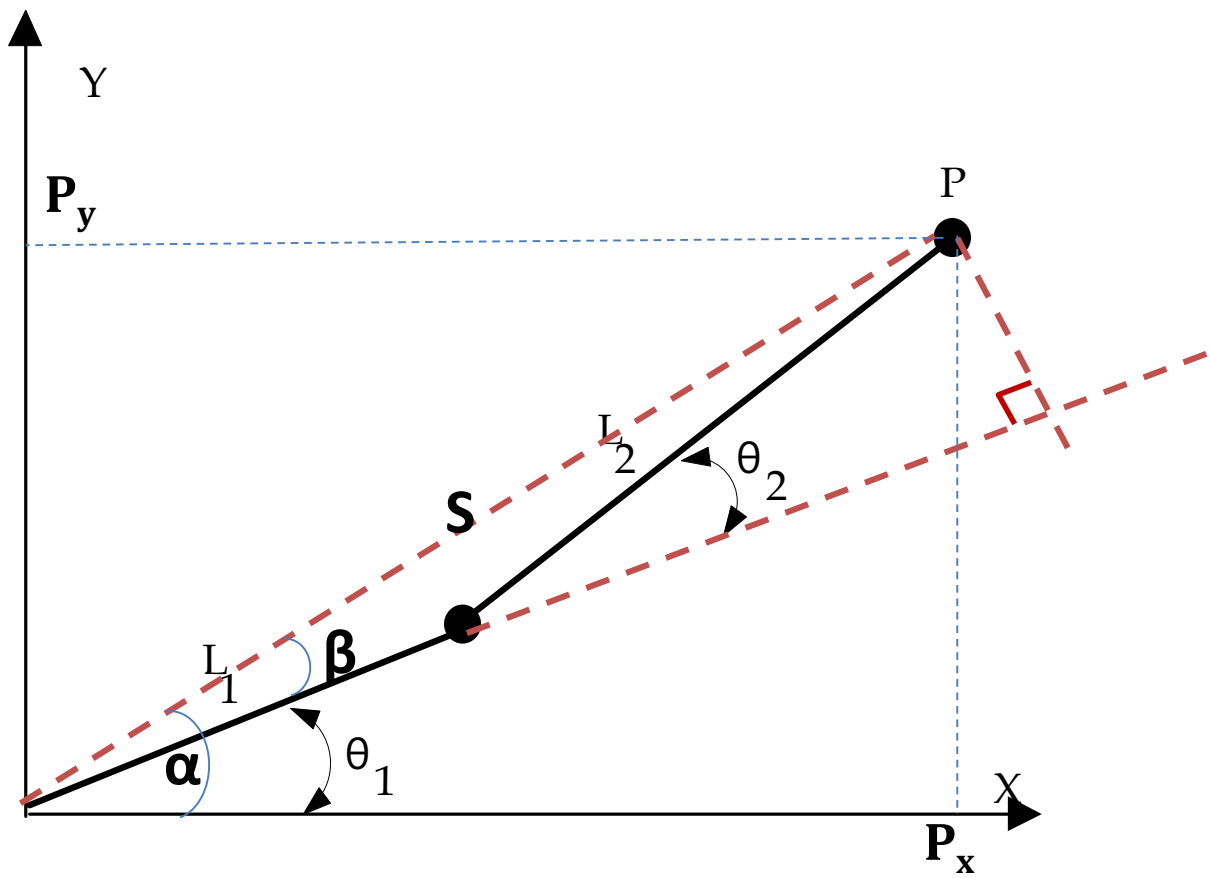






$$S = \sqrt{P_x^2 + P_y^2}$$

$$\alpha = \tan^{-1} \frac{P_y}{P_x}$$



$$S^2 = (l_1 + l_2 \cos \theta_2)^2 + (l_2 \sin \theta_2)^2$$

$$= l_1^2 + 2l_1 l_2 \cos \theta_2 + l_2^2 \cos^2 \theta_2 + l_2^2 \sin^2 \theta_2$$

$$= l_1^2 + 2l_1 l_2 \cos \theta_2 + l_2^2$$

$$S^2 = l_1^2 + 2l_1l_2\cos\theta_2 + l_2^2$$

$$\cos\theta_2 = \frac{S^2 - l_1^2 - l_2^2}{2l_1l_2} \quad \sin\theta_2 = \sqrt{1 - \cos^2\theta_2}$$

$$\theta_2 = \tan^{-1} \frac{\sin\theta_2}{\cos\theta_2}$$

For elbow up  $\theta_2 = +ve$

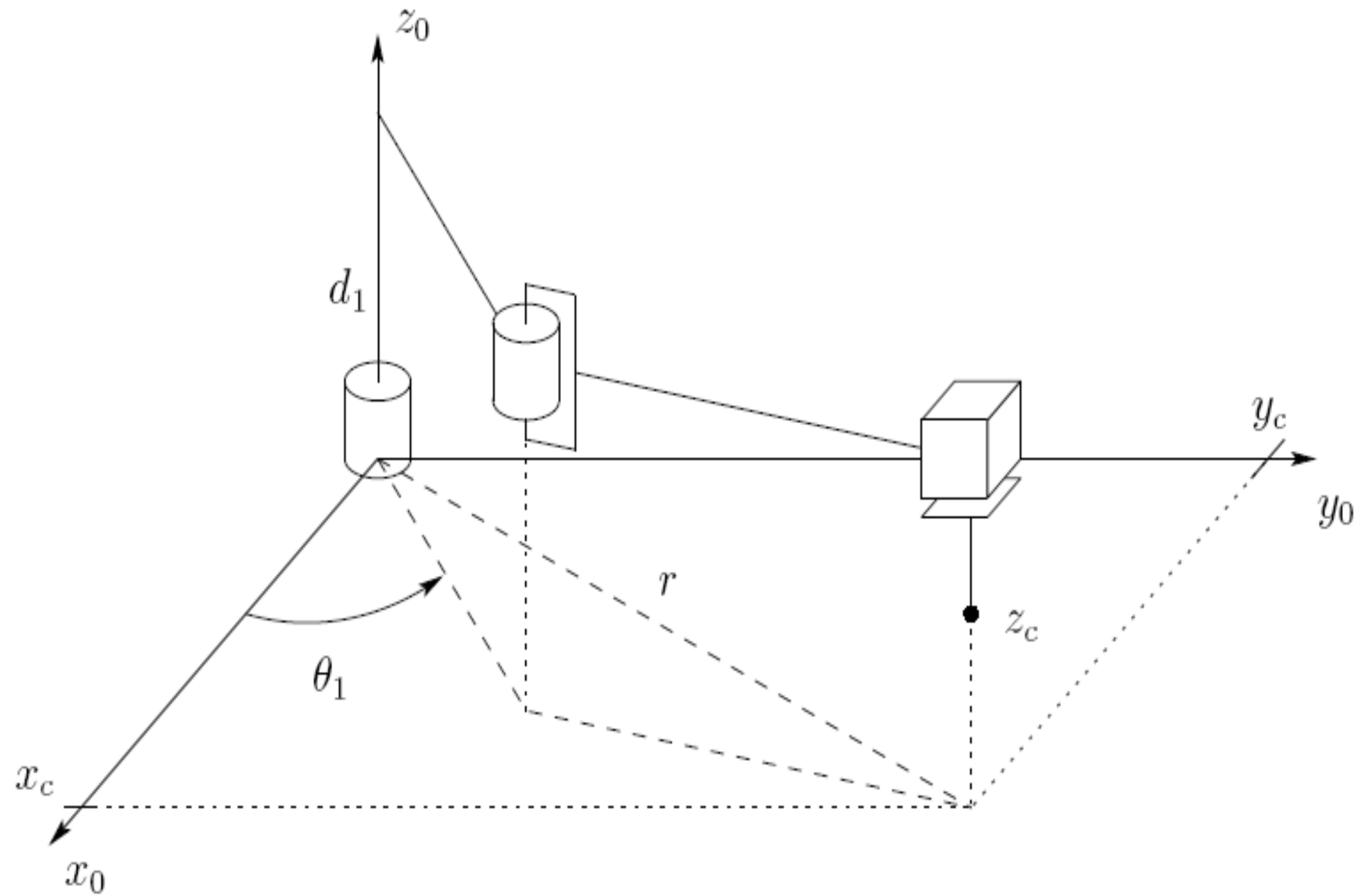
For elbow down  $\theta_2 = -ve$

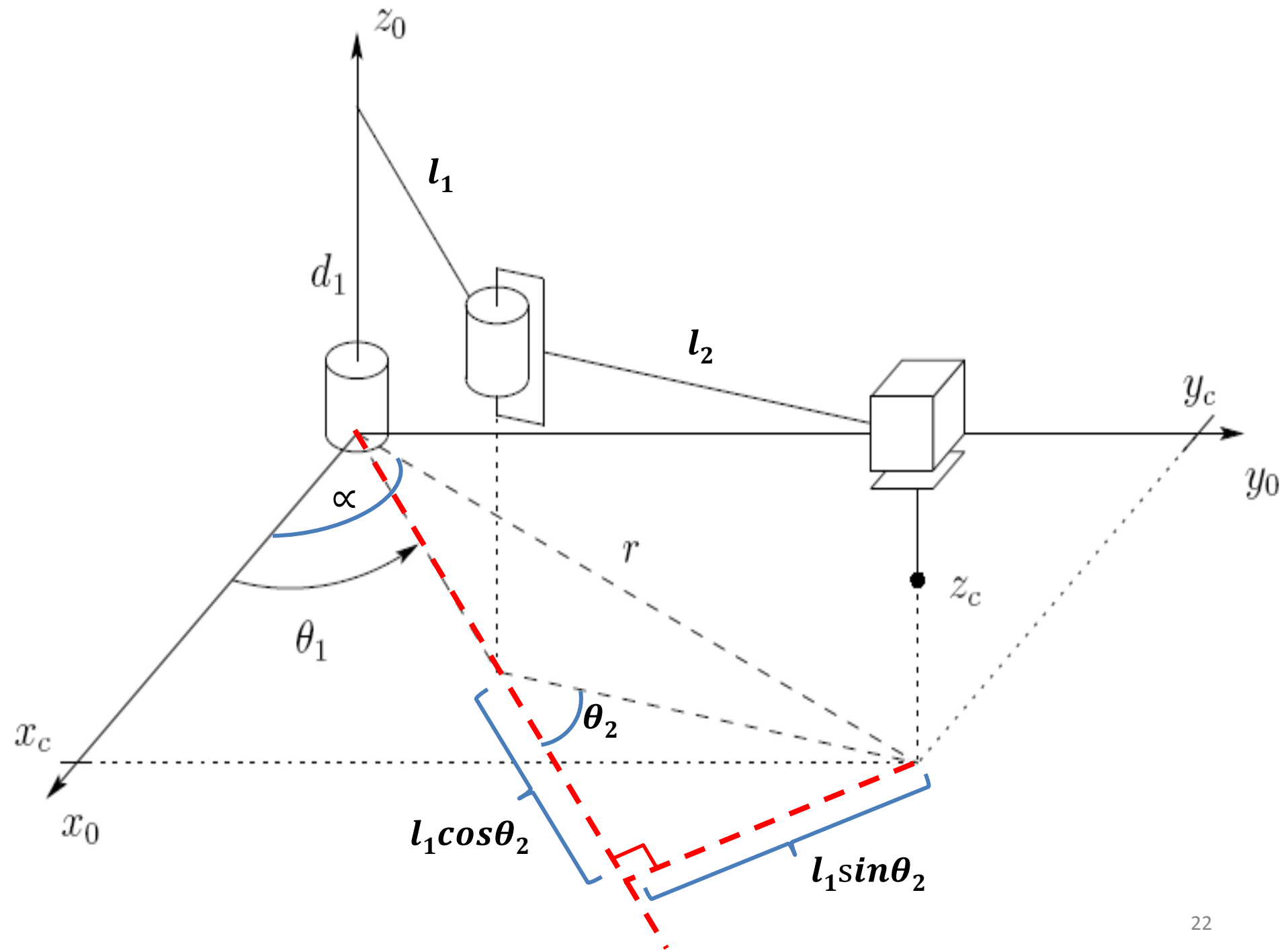
$$\tan\beta = \frac{l_2\sin\theta_2}{l_1 + l_2\cos\theta_2} \quad \beta = \tan^{-1} \frac{l_2\sin\theta_2}{l_1 + l_2\cos\theta_2}$$

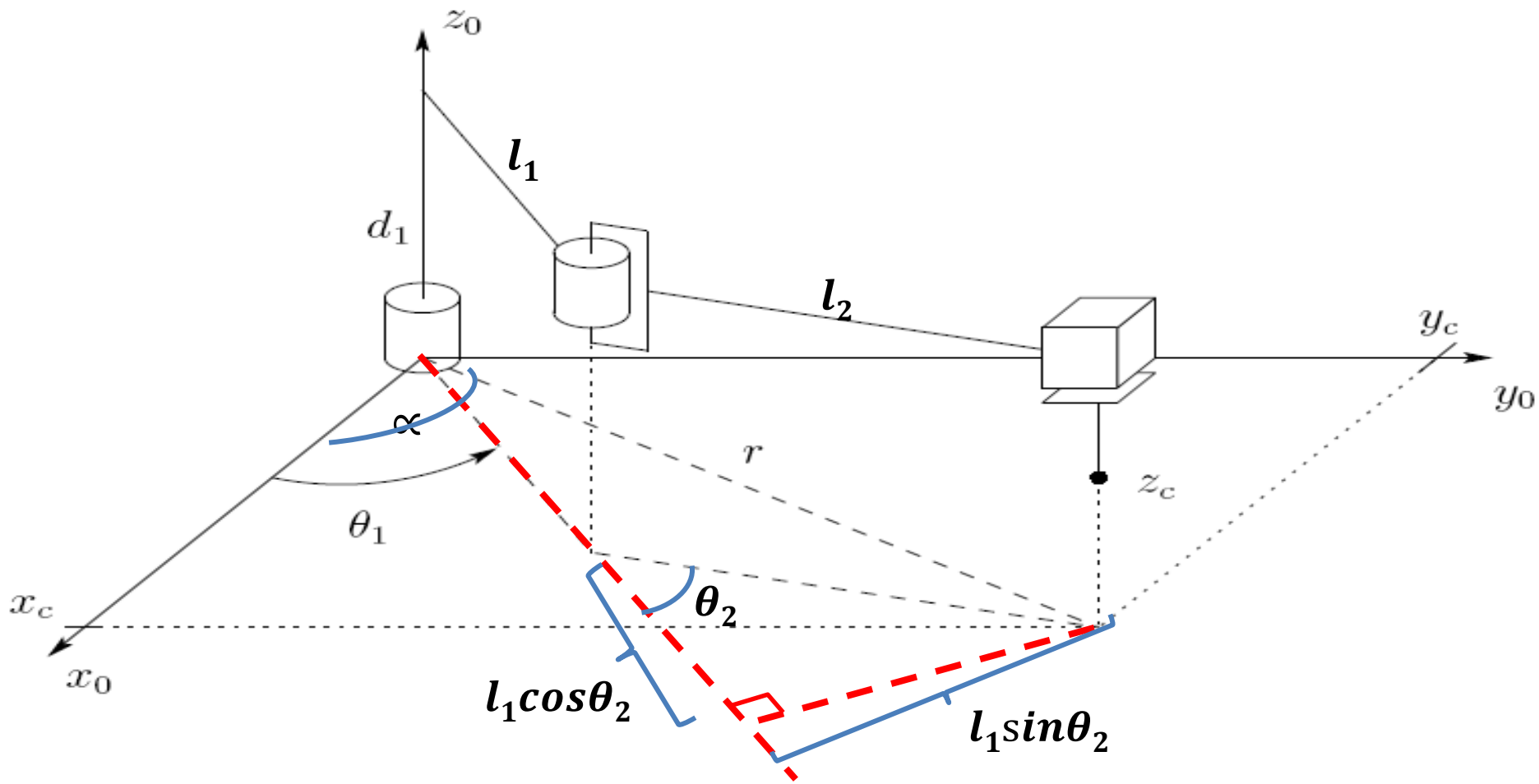
For elbow up  $\theta_1 = \alpha - \beta$

For elbow down  $\theta_1 = \alpha + \beta$

# Example







$$\begin{aligned}
 r^2 &= P_x^2 + P_y^2 = (l_1 + l_2 \cos \theta_2)^2 + (l_2 \sin \theta_2)^2 \\
 &= l_1^2 + 2l_1 l_2 \cos \theta_2 + l_2^2 \cos^2 \theta_2 + l_2^2 \sin^2 \theta_2 \\
 &= l_1^2 + 2l_1 l_2 \cos \theta_2 + l_2^2
 \end{aligned}$$

$$r^2 = l_1^2 + 2l_1l_2\cos\theta_2 + l_2^2$$

$$\cos\theta_2 = \frac{r^2 - l_1^2 - l_2^2}{2l_1l_2} \quad \sin\theta_2 = \sqrt{1 - \cos^2\theta_2}$$

$$\theta_2 = \tan^{-1} \frac{\sin\theta_2}{\cos\theta_2}$$

$$\alpha = \tan^{-1} \frac{P_y}{P_x}$$

$$\tan\beta = \frac{l_2\sin\theta_2}{l_1 + l_2\cos\theta_2} \quad \beta = \tan^{-1} \frac{l_2\sin\theta_2}{l_1 + l_2\cos\theta_2}$$

For elbow up  $\theta_1 = \alpha - \beta$

For elbow down  $\theta_1 = \alpha + \beta$



*End of Lec.*