Robotics

Lecture 6

Inverse Kinematics

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Inverse Kinematics (IK)

"Given a goal position find the joint angles for the robot arm"

Inverse Kinematics

- The inverse kinematics is needed in the control of manipulators.
- Solving the inverse kinematics is computationally expansive and generally takes a very long time in the real time control of manipulators.
- IK generally harder than FK
- Sometimes no analytical solution
- Sometimes multiple solutions
- Sometimes no solution
  - Outside workspace

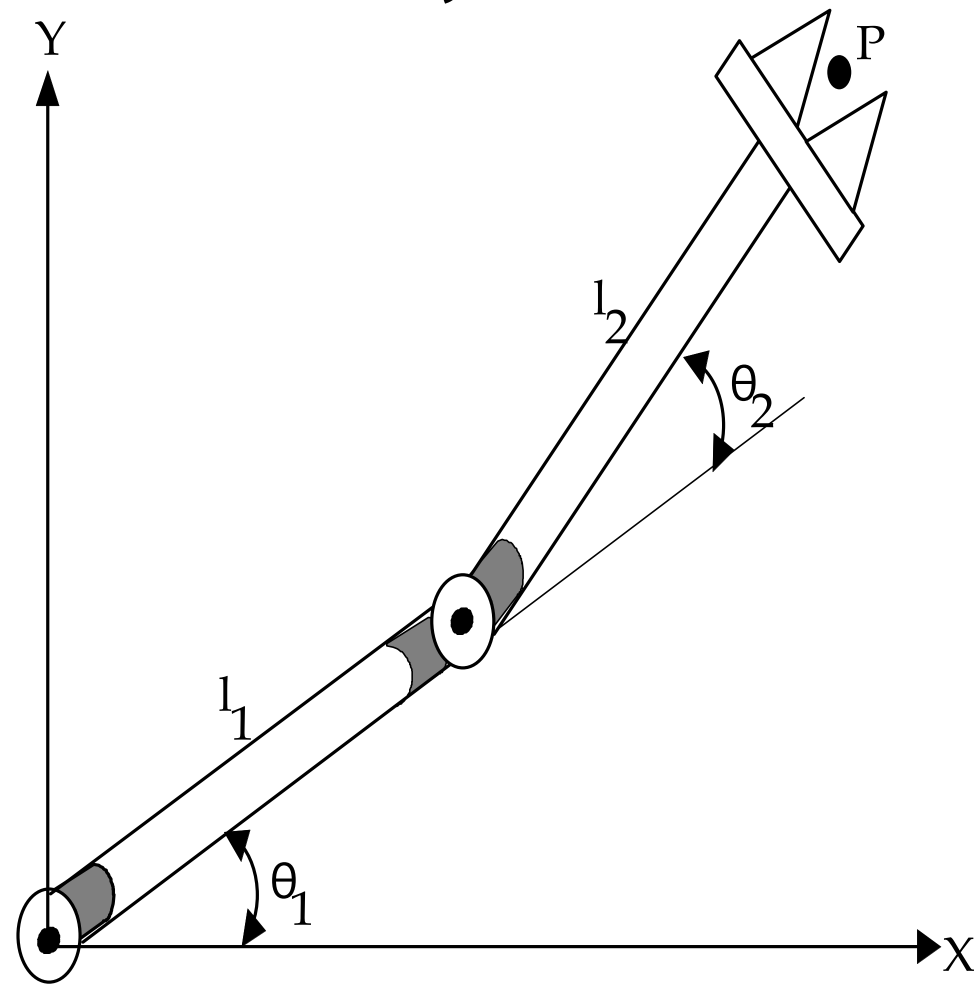
Inverse kinematics

- **Analytical Method**
  Joint variables solved according to given configuration data
  - **Geometric solution**
    For simple structures,2-DOF
  - **Algebraic solution**
    For more links and in 3 dimensions
- **Numerical Method**
  Joint variables obtained by numerical techniques

Geometric Solution Approach

Geometric Solution Approach

- It is applied to the simple robot structures, such as, 2-DOF planer manipulator whose joints are both revolute.



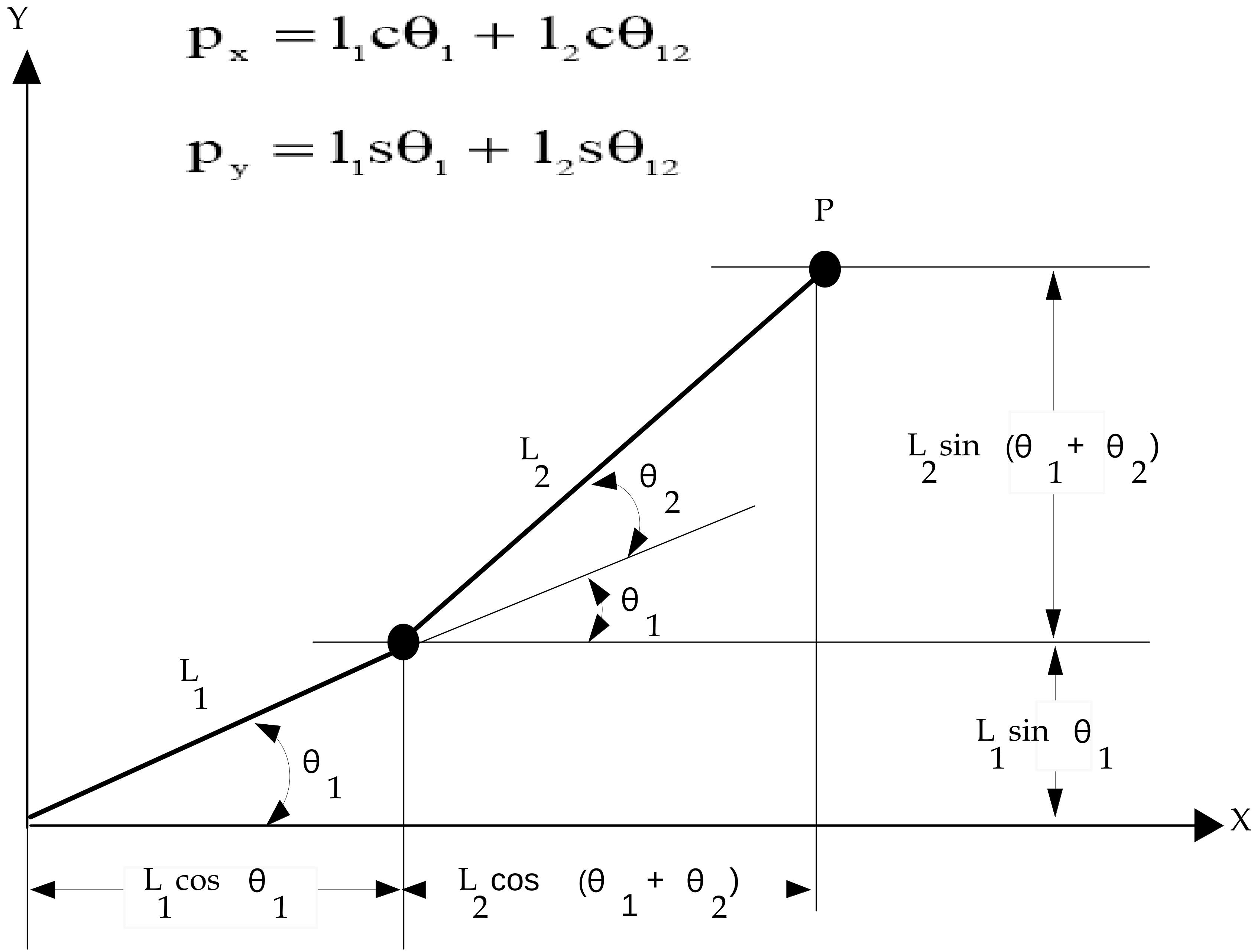
- In the shown Figure, the components of point P (p_x , p_y) are determined as follows.

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$



$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$

- The solution of θ_2 can be computed from summation of squaring both previous equations

$$p_x^2 = l_1^2 c^2\theta_1 + l_2^2 c^2\theta_{12} + 2l_1l_2 c\theta_1 c\theta_{12}$$

$$p_y^2 = l_1^2 s^2\theta_1 + l_2^2 s^2\theta_{12} + 2l_1l_2 s\theta_1 s\theta_{12}$$

$$p_x^2 + p_y^2 = l_1^2(c^2\theta_1 + s^2\theta_1) + l_2^2(c^2\theta_{12} + s^2\theta_{12}) + 2l_1l_2(c\theta_1 c\theta_{12} + s\theta_1 s\theta_{12})$$

- Since $c^2\theta_1 + s^2\theta_1 = 1$, the equation is simplified as:

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2(c\theta_1[c\theta_1 c\theta_2 - s\theta_1 s\theta_2] + s\theta_1[s\theta_1 c\theta_2 + c\theta_1 s\theta_2])$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2(c^2\theta_1 c\theta_2 - c\theta_1 s\theta_1 s\theta_2 + s^2\theta_1 c\theta_2 + c\theta_1 s\theta_1 s\theta_2)$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2(c\theta_2[c^2\theta_1 + s^2\theta_1])$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2 c\theta_2$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2c\theta_2$$

and so

$$c\theta_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

Since, $c^2\theta_i + s^2\theta_i = 1$ (i =1,2,3,……), $s\theta_2$ is obtained as

$$s\theta_2 = \pm\sqrt{1 - \left(\frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)^2}$$

Finally, two possible solutions for θ_2 can be written as

$$\theta_2 = A\tan 2\left(\pm\sqrt{1 - \left(\frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)^2}, \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)$$

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12} \quad ....1$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12} \quad ....2$$

- The solution of θ_1 multiply each side of equation 1 by $c\theta_1$ and equation 2 by $s\theta_1$ and add the resulting equations in order to find the solution of θ_1 in terms of link parameters and the known variable θ_2.

$$c\theta_1 p_x = l_1 c^2\theta_1 + l_2 c^2\theta_1 c\theta_2 - l_2 c\theta_1 s\theta_1 s\theta_2$$

$$s\theta_1 p_y = l_1 s^2\theta_1 + l_2 s^2\theta_1 c\theta_2 + l_2 s\theta_1 c\theta_1 s\theta_2$$

$$c\theta_1 p_x + s\theta_1 p_y = l_1 (c^2\theta_1 + s^2\theta_1) + l_2 c\theta_2 (c^2\theta_1 + s^2\theta_1)$$

The simplified equation obtained as follows.

$$c\theta_1 p_x + s\theta_1 p_y = l_1 + l_2 c\theta_2 \quad ....3$$

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12} \quad \text{........1}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12} \quad \text{........2}$$

- Multiply each side of equation 1 by $-s\theta_1$ and equation 2 by $c\theta_1$ and add the resulting equations

$$-s\theta_1 p_x = -l_1 s\theta_1 c\theta_1 - l_2 s\theta_1 c\theta_1 c\theta_2 + l_2 s^2\theta_1 s\theta_2$$

$$c\theta_1 p_y = l_1 s\theta_1 c\theta_1 + l_2 c\theta_1 s\theta_1 c\theta_2 + l_2 c^2\theta_1 s\theta_2$$

$$-s\theta_1 p_x + c\theta_1 p_y = l_2 s\theta_2 (c^2\theta_1 + s^2\theta_1)$$

The simplified equation is given by

$$-s\theta_1 p_x + c\theta_1 p_y = l_2 s\theta_2 \quad \text{...........4}$$

Now, multiply each side of equation 3 by p_x and equation 4 by p_y and add the resulting equations in order to obtain $c\theta_1$.

$$c\theta_1 p_x^2 + s\theta_1 p_x p_y = p_x(l_1 + l_2 c\theta_2)$$
$$-s\theta_1 p_x p_y + c\theta_1 p_y^2 = p_y l_2 s\theta_2$$
$$c\theta_1(p_x^2 + p_y^2) = p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2$$

and so

$$c\theta_1 = \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}$$

$$c\theta_1 = \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}$$

$s\theta_1$ is obtained as

$$s\theta_1 = \pm\sqrt{1 - \left(\frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right)^2}$$

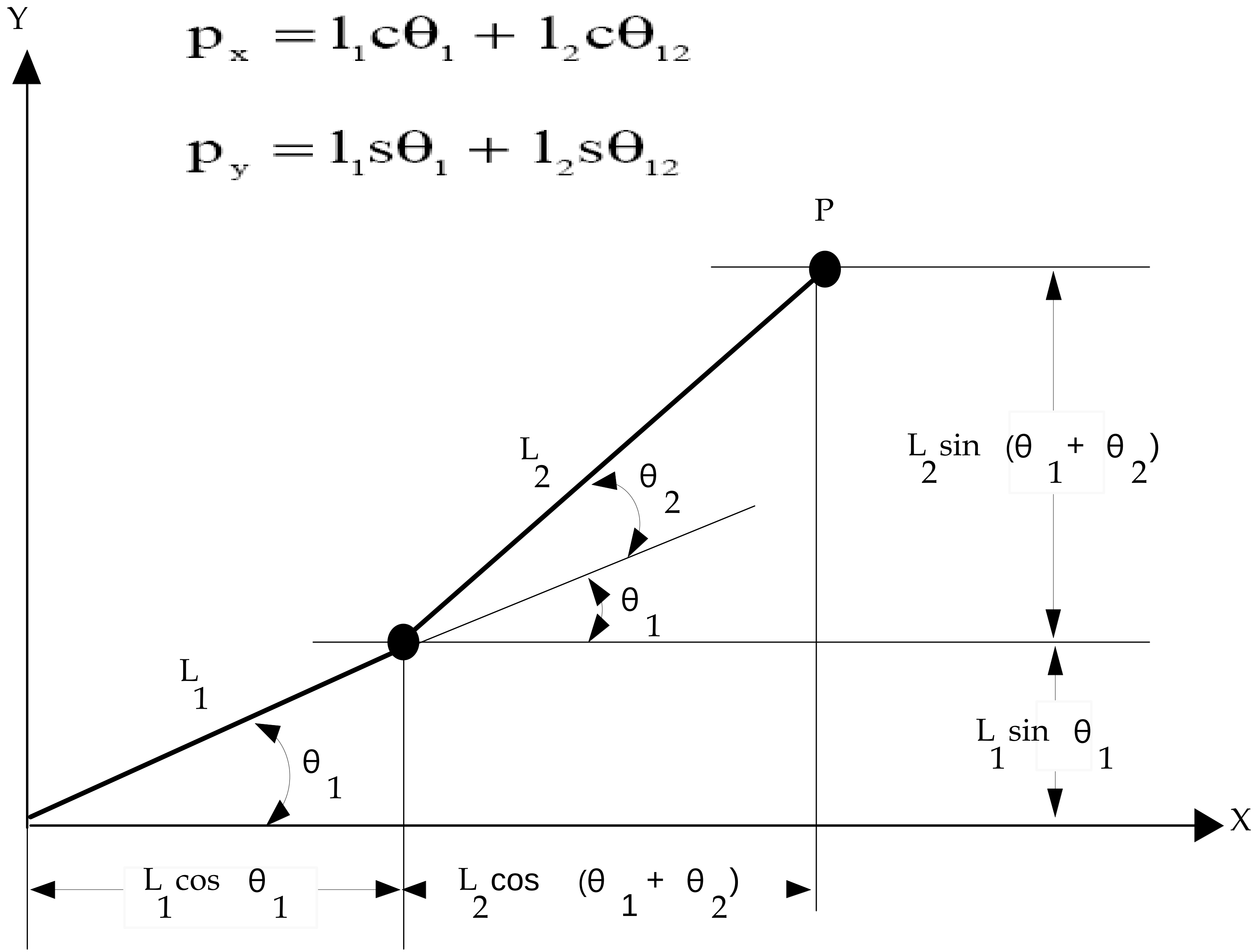
As a result, two possible solutions for θ_1 can be written

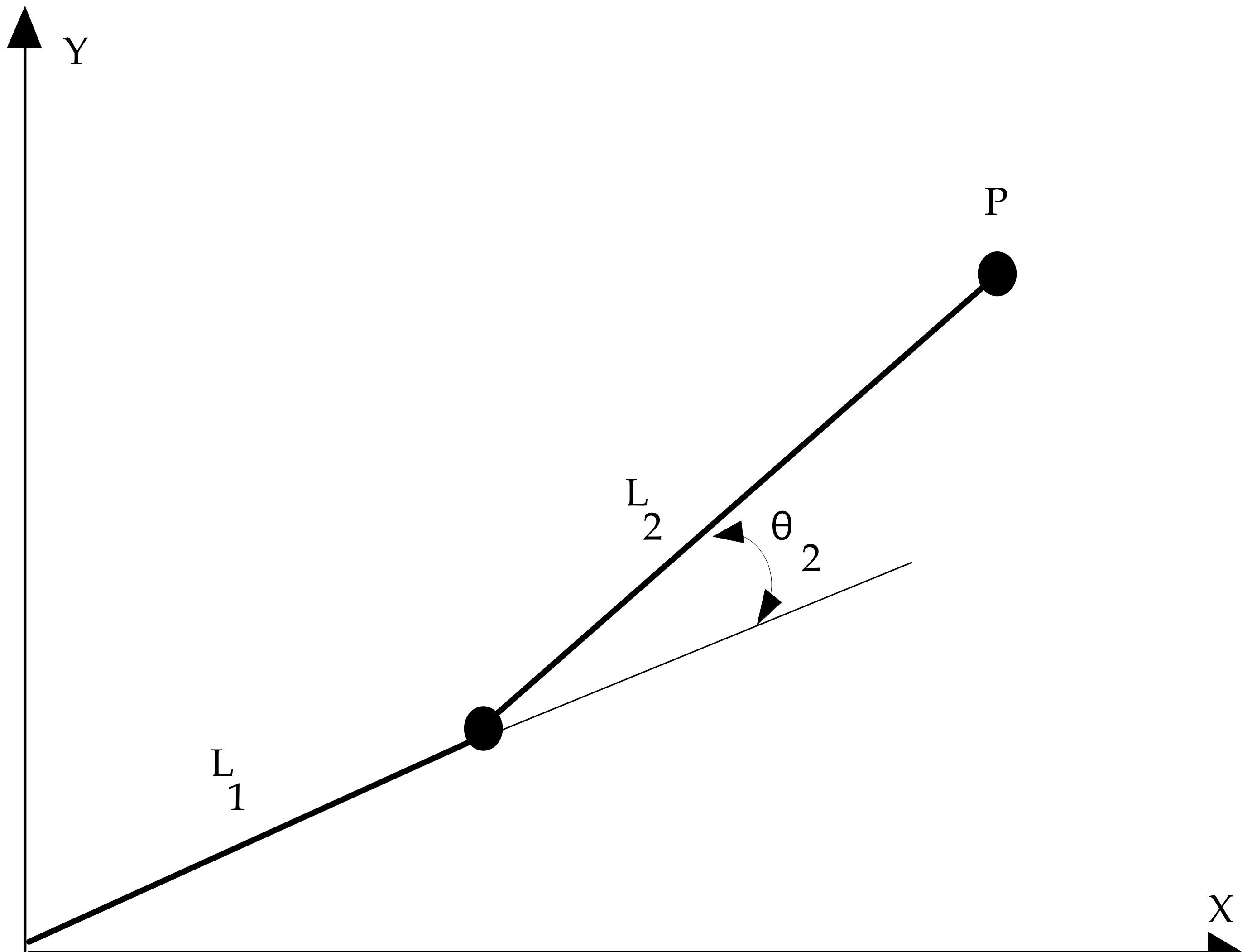
$$\theta_1 = A\tan 2\left(\pm\sqrt{1 - \left(\frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right)^2}, \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right)$$

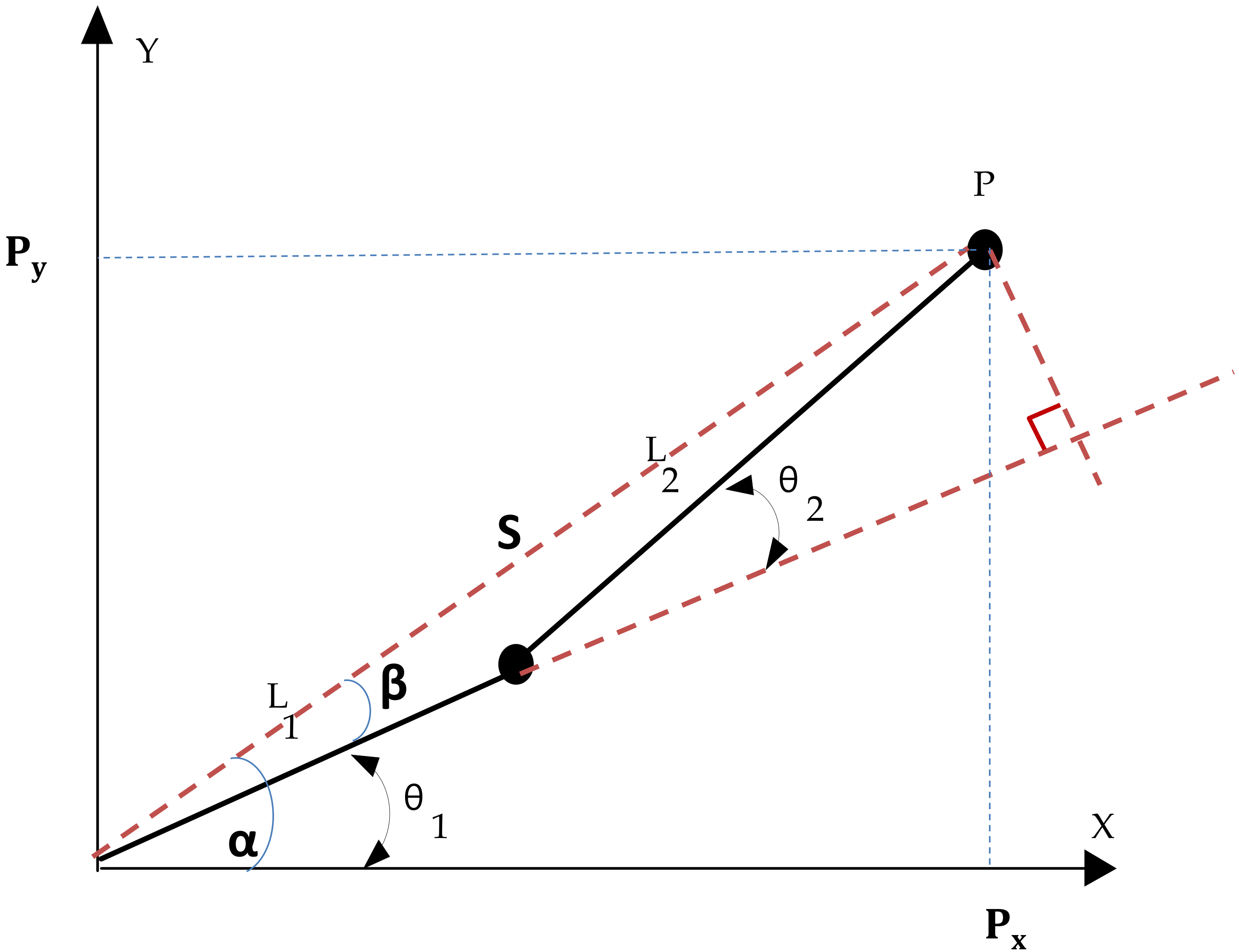
Another solution

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$



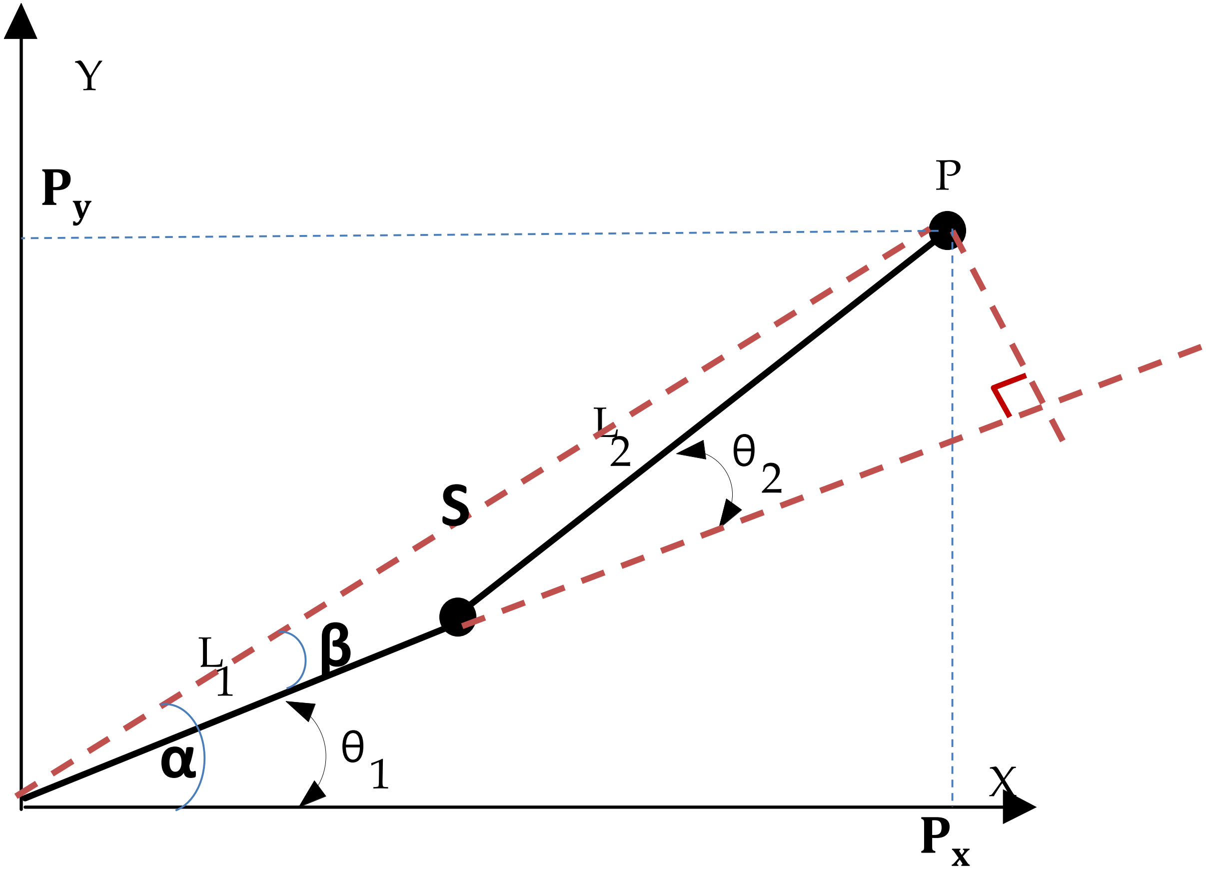

Y
P
L
2
θ
2
L
1
X



Y
P
P_y
L_2
θ_2
S
β
L_1
θ_1
α
X
P_x

$$S = \sqrt{P_x^2 + P_y^2}$$

$$\propto = \tan^{-1}\frac{P_y}{P_x}$$



$$S^2 = (l_1 + l_2 cos\ \theta_2)^2 + (l_2 sin\theta_2)^2$$

$$= {l_1}^2 + 2l_1 l_2 cos\ \theta_2 + {l_2}^2 cos^2\theta_2 + {l_2}^2 sin^2\theta_2$$

$$= {l_1}^2 + 2l_1 l_2 cos\ \theta_2 + {l_2}^2$$

$$S^2 = {l_1}^2 + 2l_1 l_2 cos\ \theta_2 + {l_2}^2$$

$$cos\ \theta_2 = \frac{S^2 - {l_1}^2 - {l_2}^2}{2l_1 l_2} \qquad sin\theta_2 = \sqrt{1 - cos^2\theta_2}$$

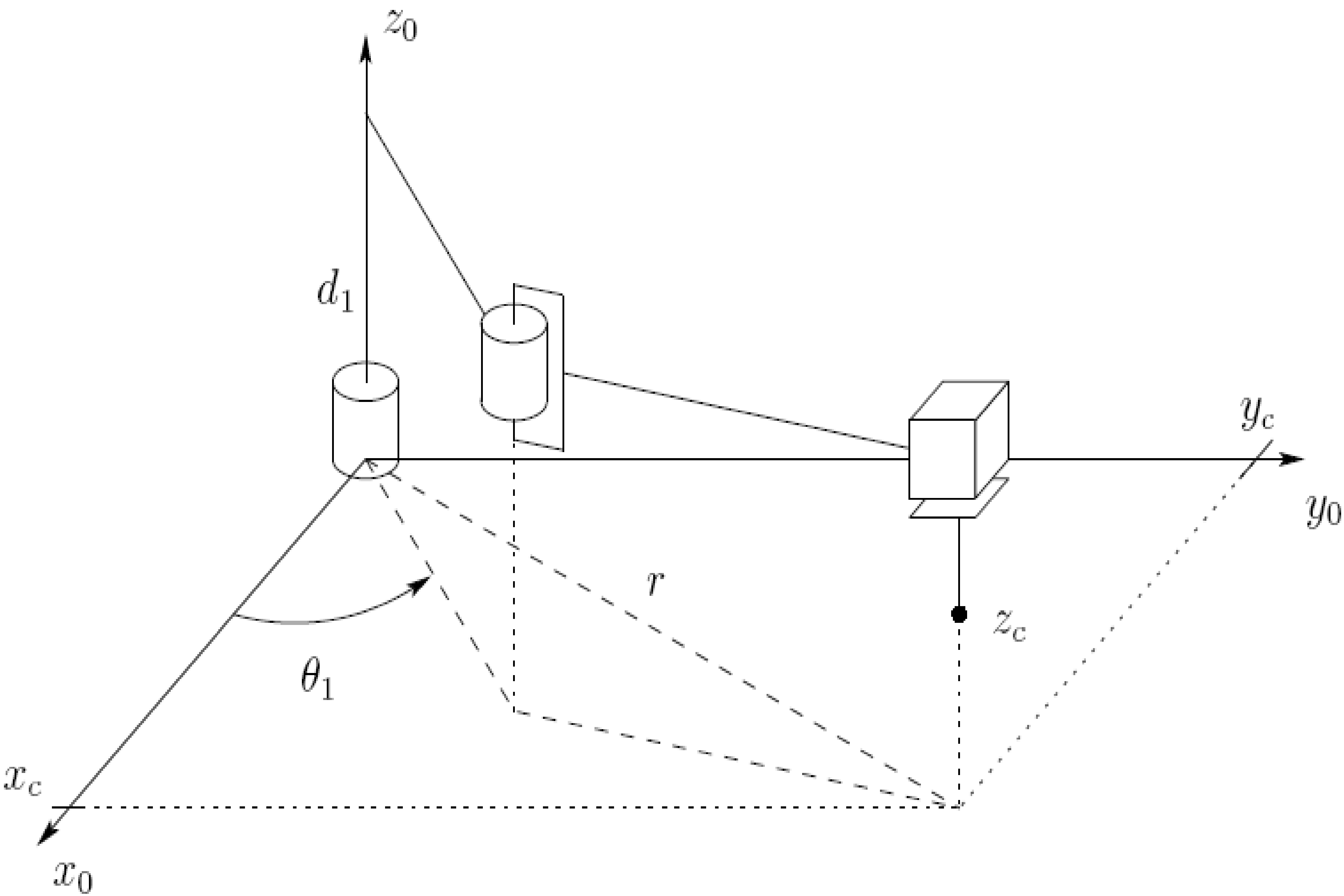
$$\theta_2 = \tan^{-1}\frac{sin\theta_2}{cos\ \theta_2}$$

For elbow up $\theta_2 = +ve$ For elbow down $\theta_2 = -ve$

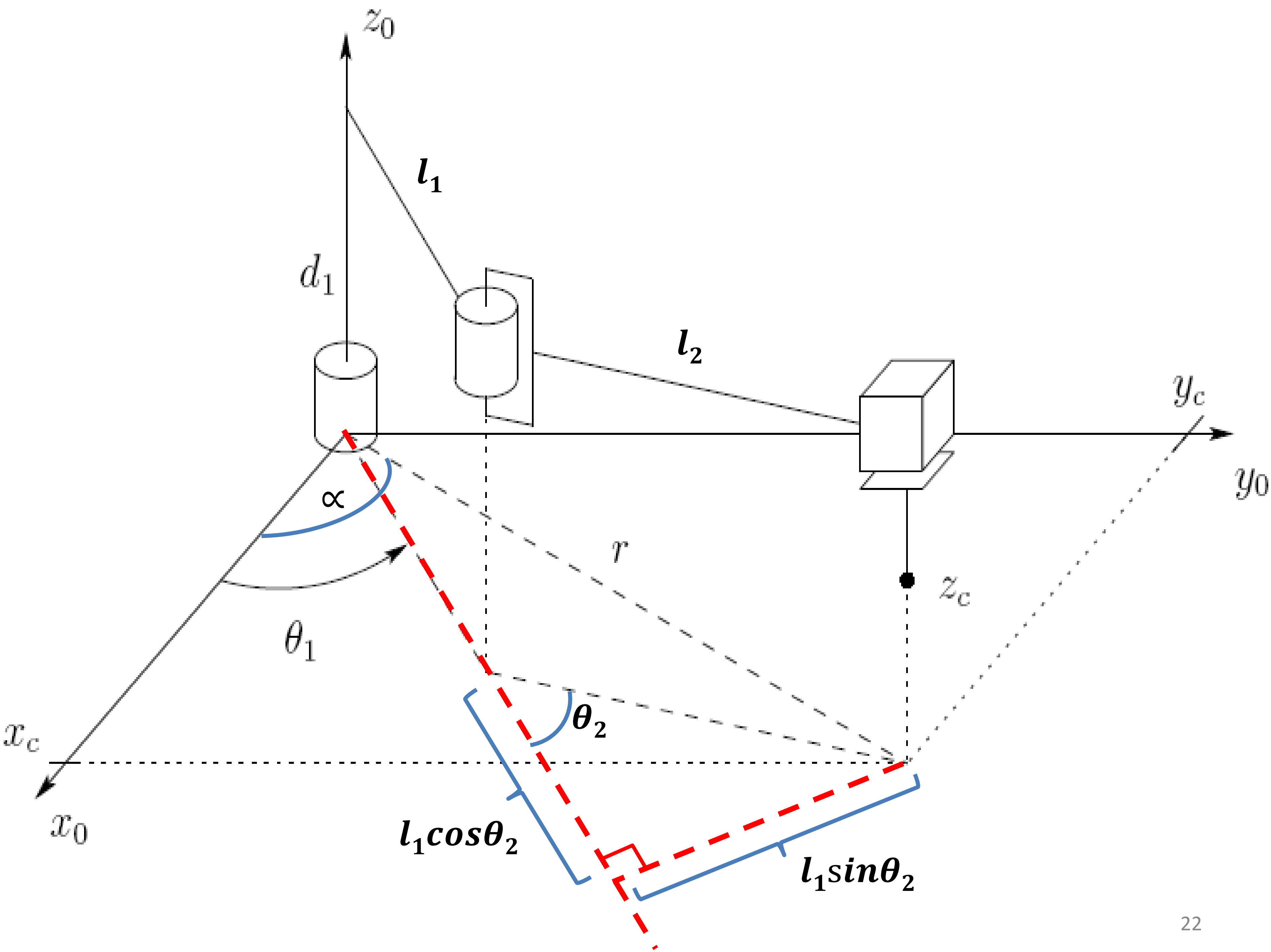
$$\tan\beta = \frac{l_2\, sin\theta_2}{l_1 + l_2\, cos\ \theta_2} \qquad \beta = \tan^{-1}\frac{l_2\, sin\theta_2}{l_1 + l_2\, cos\ \theta_2}$$

For elbow up $\theta_1 = \propto - \beta$ For elbow down $\theta_1 = \propto + \beta$

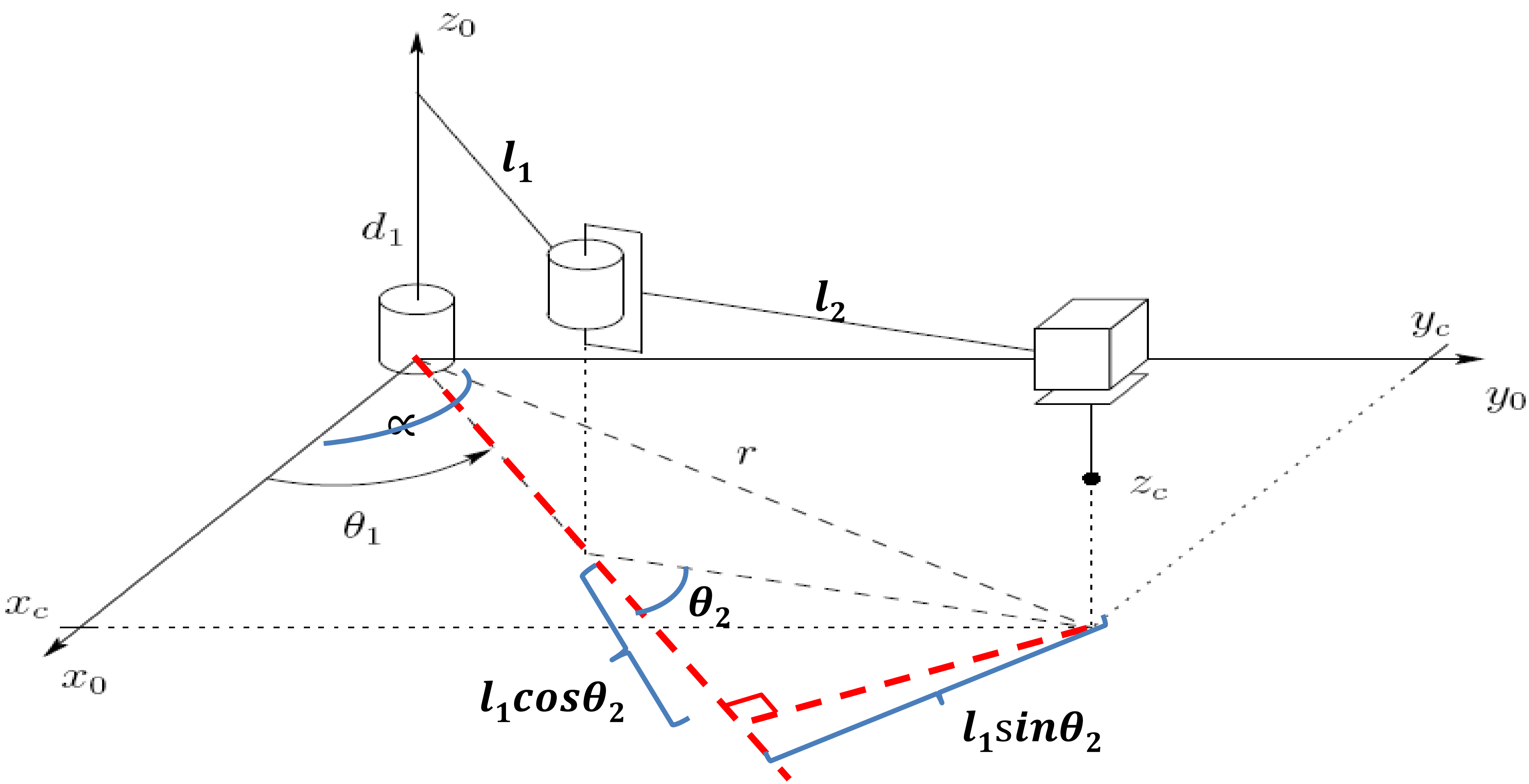
Example



z_0
d_1
y_c
y_0
r
z_c
θ_1
x_c
x_0



z_0
l_1
d_1
l_2
y_c
y_0
$\propto$
r
z_c
θ_1
θ_2
x_c
x_0
$l_1 cos\theta_2$
$l_1 sin\theta_2$



$$r^2 = P_x^2 \ + P_y^2 = (l_1 + l_2 cos\ \theta_2)^2 + (l_2 sin\theta_2)^2$$

$$= {l_1}^2 + 2 l_1 l_2 cos\ \theta_2 + {l_2}^2 cos^2\theta_2 + {l_2}^2 sin^2\theta_2$$

$$= {l_1}^2 + 2 l_1 l_2 cos\ \theta_2 + {l_2}^2$$

$$r^2 = l_1{}^2 + 2l_1l_2 cos\ \theta_2 + l_2{}^2$$

$$cos\ \theta_2 = \frac{r^2 - l_1{}^2 - l_2{}^2}{2l_1l_2} \qquad sin\theta_2 = \sqrt{1 - cos^2\theta_2}$$

$$\theta_2 = \tan^{-1}\frac{sin\theta_2}{cos\ \theta_2}$$

$$\propto = \tan^{-1}\frac{P_y}{P_x}$$

$$\tan\beta = \frac{l_2\, sin\theta_2}{l_1 + l_2\, cos\ \theta_2} \qquad \beta = \tan^{-1}\frac{l_2\, sin\theta_2}{l_1 + l_2\, cos\ \theta_2}$$

For elbow up $\theta_1 = \propto - \beta$ For elbow down $\theta_1 = \propto + \beta$

End of Lec.