Computer Control of Dynamic Systems

Lecture-1

Introduction

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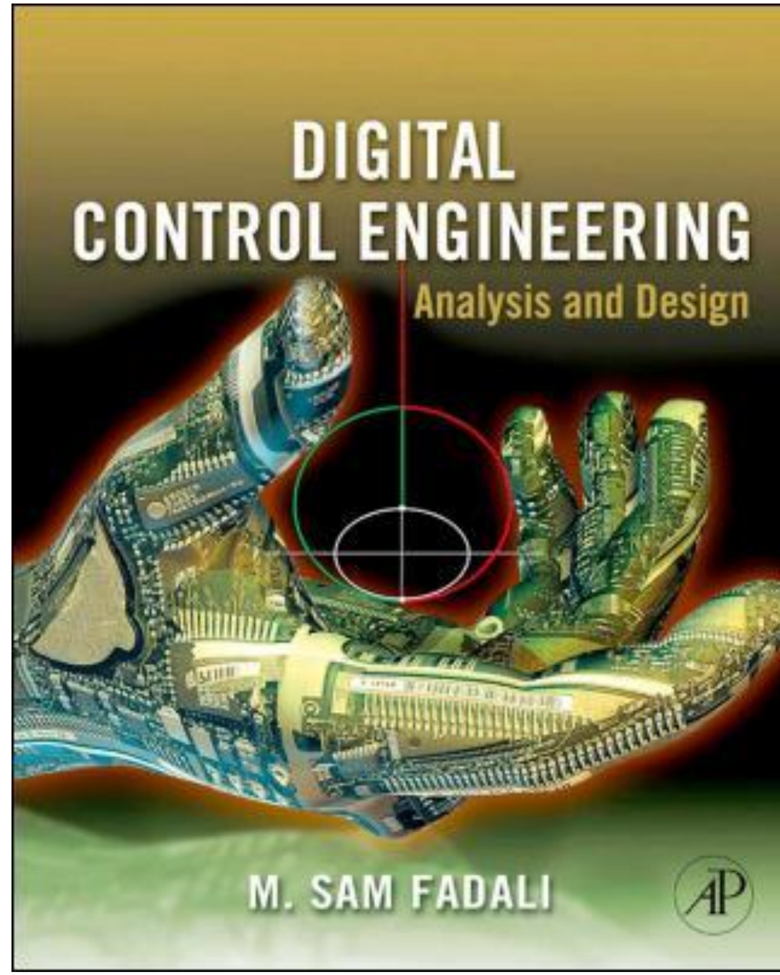
Recommended Book

- M.S. Fadali, “Digital Control Engineering – Analysis and Design”, Elsevier, 2009. ISBN: 13: 978-0-12-374498-2



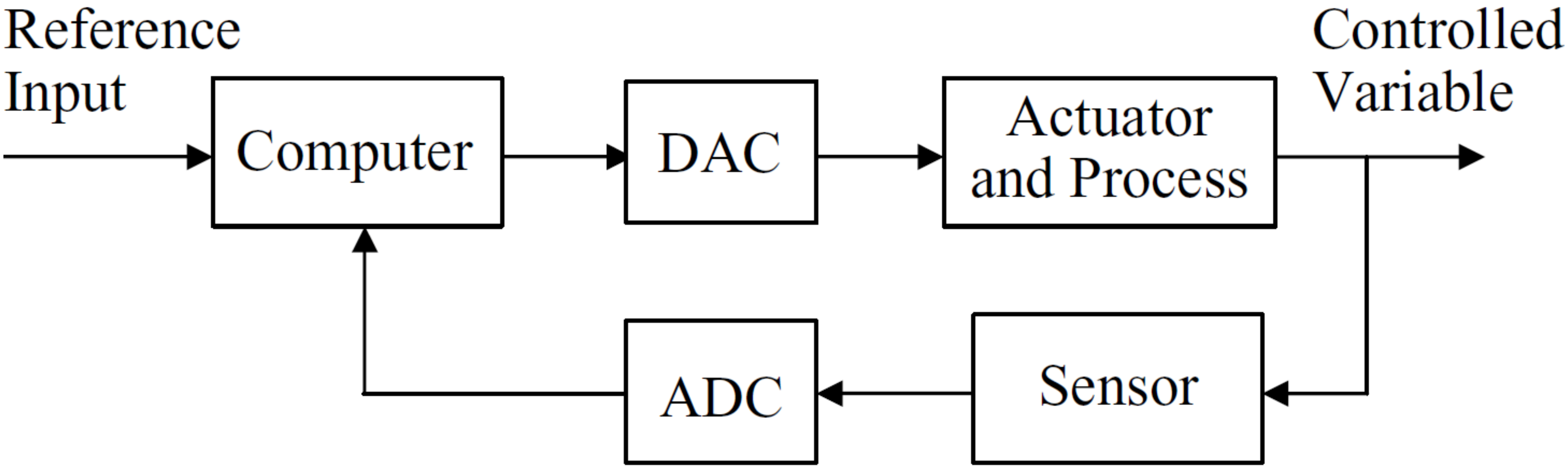
Introduction

- All control systems are based on computer control.
- It is therefore important to understand computer-controlled systems.
- The computers are often used in embedded systems.
- An embedded system is a built in computer/microprocessor that is a part of a larger system.
- For example, vehicles, home electronics, cellular telephones, and standalone controllers.

Introduction cont.

- Computer-controlled systems
  - More adaptable.
  - More flexible to Modify without complete replacement.
  - More network security.
  - More Accurate.
  - More cost effective than analogue controllers.

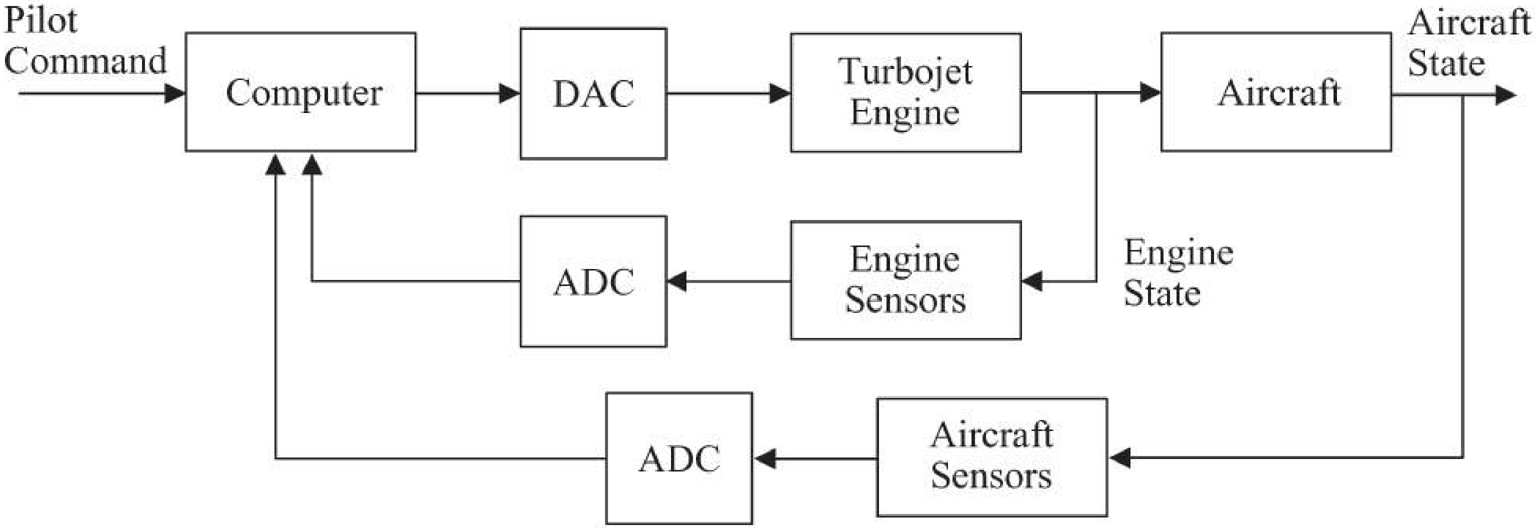
Structure of a Computer-Controlled System



Schematic diagram of a computer-controlled system.

Examples of computer-controlled Systems

Aircraft Turbojet Engine



System Representation

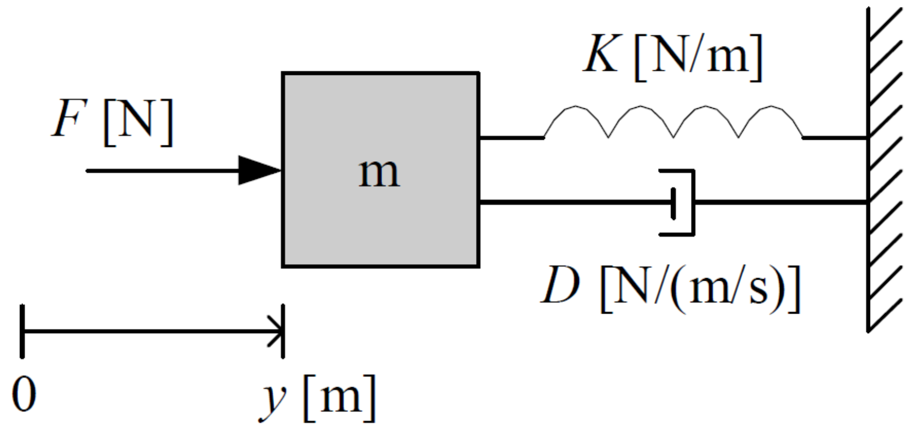
Differential Equation vs Difference Equation

- A differential equation expresses the change in a variable as a result of an *infinitesimal* change in another variable.

- A difference equation expresses the change in a variable as a result of a *finite* change in another variable.

Differential Equation

- Following figure shows a mass-spring-damper-system. Where y is position, F is applied force D is damping constant and K is spring constant.



$$F(t) = m\ddot{y}(t) + D\dot{y}(t) + Ky(t)$$

- Rearranging above equation in following form

$$\ddot{y}(t) = \frac{1}{m}F(t) - \frac{D}{m}\dot{y}(t) - \frac{K}{m}y(t)$$

Differential Equation

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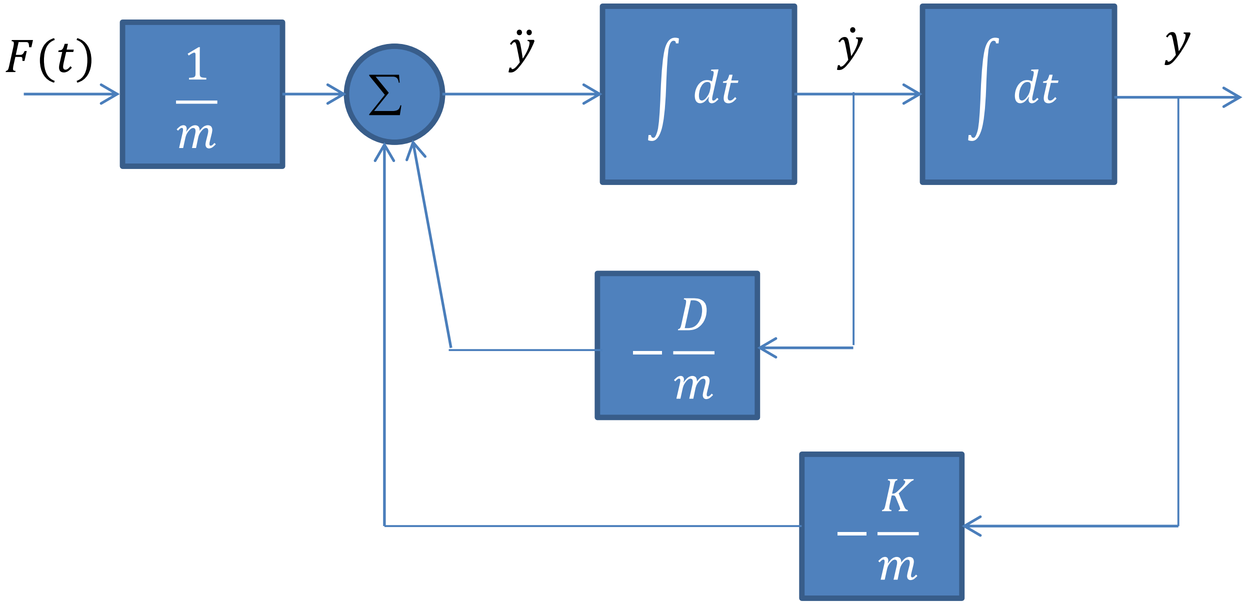
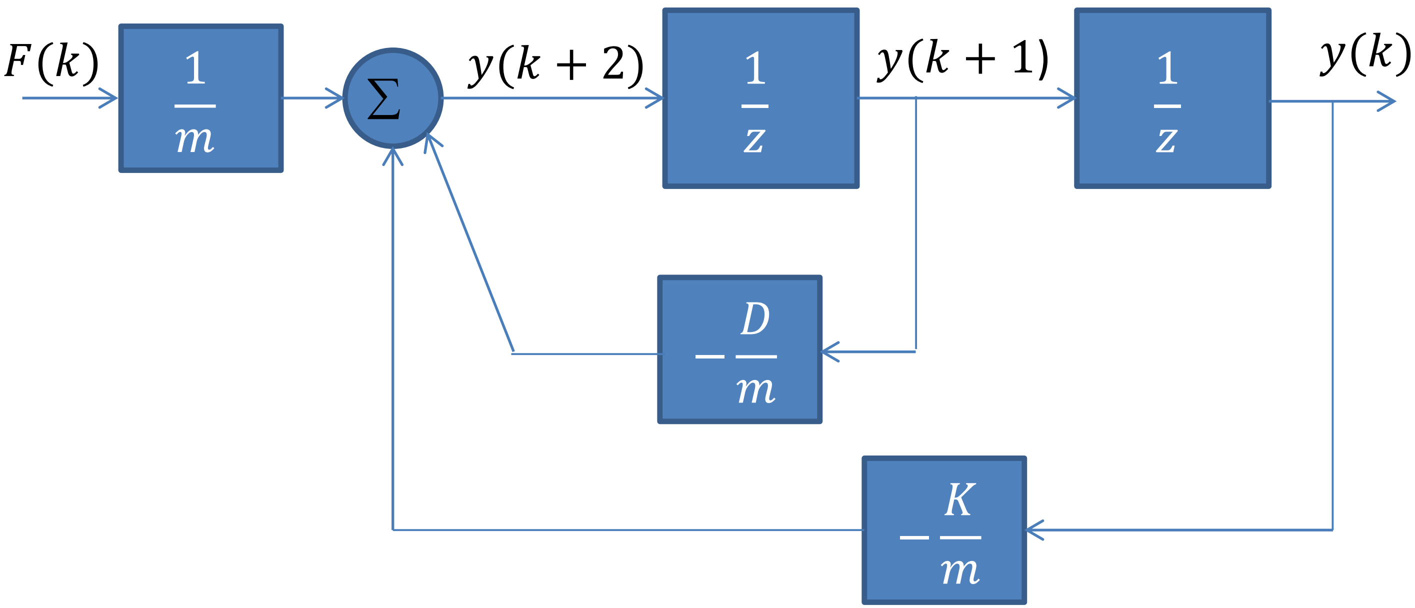
- Rearranging above equation in following form



Difference Equation

$$y(k+2) = \frac{1}{m}F(k) - \frac{D}{m}y(k+1)\ \frac{K}{m}y(k)$$



Difference Equations

- General form

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k)$$
$$= b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)$$

- Where coefficients a_{n-1}, a_{n-2},… and b_n, b_{n-1},… are constant.
- $u(k)$ is forcing function

State Space Modelling

State Space Modeling

- State space equations can be simplified as

$$\dot{x}(t) = Ax(t) + Bu(t)$$

State Equation

$$y(t) = Cx(t) + Du(t)$$

Output Equation

System type	State-space model
Continuous time-invariant	$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$ $\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$
Continuous time-variant	$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$ $\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$
Explicit discrete time-invariant	$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$ $\mathbf{y}(k) = C\mathbf{x}(k) + D\mathbf{u}(k)$
Explicit discrete time-variant	$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$ $\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k)$
Laplace domain of continuous time-invariant	$\mathbf{Y}(s) = C\mathbf{X}(s) + D\mathbf{U}(s)$ $s\mathbf{X}(s) = A\mathbf{X}(s) + B\mathbf{U}(s)$
Z-domain of discrete time-invariant	$z\mathbf{X}(z) = A\mathbf{X}(z) + B\mathbf{U}(z)$ $\mathbf{Y}(z) = C\mathbf{X}(z) + D\mathbf{U}(z)$

Example-1

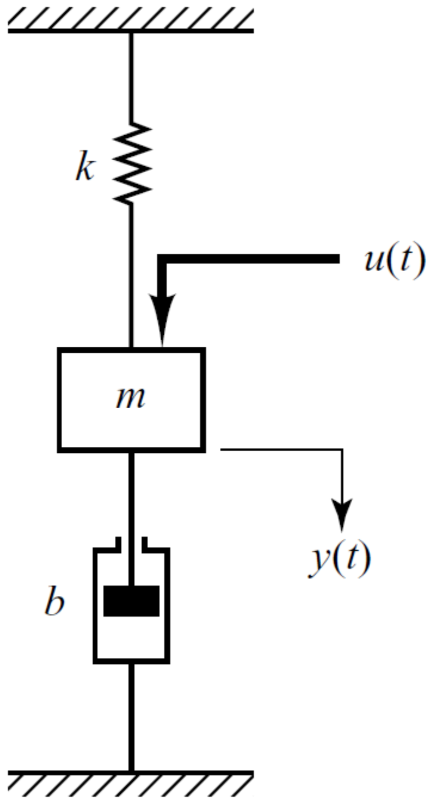
- Consider the mechanical system shown in figure. We assume that the system is linear. The external force *u(t)* is the input to the system, and the displacement *y(t)* of the mass is the output. The displacement *y(t)* is measured from the equilibrium position in the absence of the external force. This system is a single-input, single-output system.

- From the diagram, the system equation is

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = u(t)$$

- This system is of second order. This means that the system involves two integrators. Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$



Example-1

$$x_1(t) = y(t) \qquad x_2(t) = \dot{y}(t) \qquad m\ddot{y}(t) + b\dot{y}(t) + ky(t) = u(t)$$

- Then we obtain

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{b}{m}\dot{y}\,(t) - \frac{k}{m}y\,(t) + \frac{1}{m}u\,(t)$$

- Or

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{b}{m}x_2\,(t) - \frac{k}{m}x_1\,(t) + \frac{1}{m}u\,(t)$$

- The output equation is

$$y(t) = x_1(t)$$

Example-1

$$\dot{x}_1(t) = x_2(t) \qquad \dot{x}_2(t) = -\frac{b}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{1}{m}u(t) \qquad y(t) = x_1(t)$$

- In a vector-matrix form,

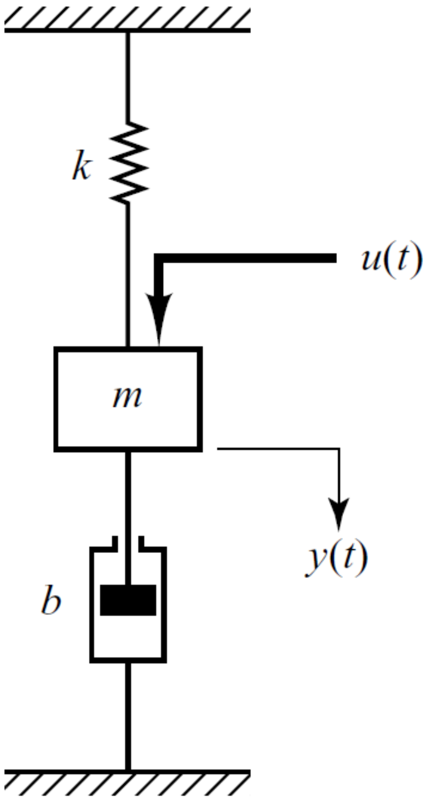
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Example-1(summary)



- The system equation is

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = u(t)$$

- Let

$$x_1(t) = y(t) \qquad x_2(t) = \dot{y}(t)$$

- Then

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{b}{m}\dot{y}\,(t) - \frac{k}{m}y\,(t) + \frac{1}{m}u\,(t)$$

- Or

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{b}{m}x_2\,(t) - \frac{k}{m}x_1\,(t) + \frac{1}{m}u\,(t)$$

$$y(t) = x_1(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u(t)$$

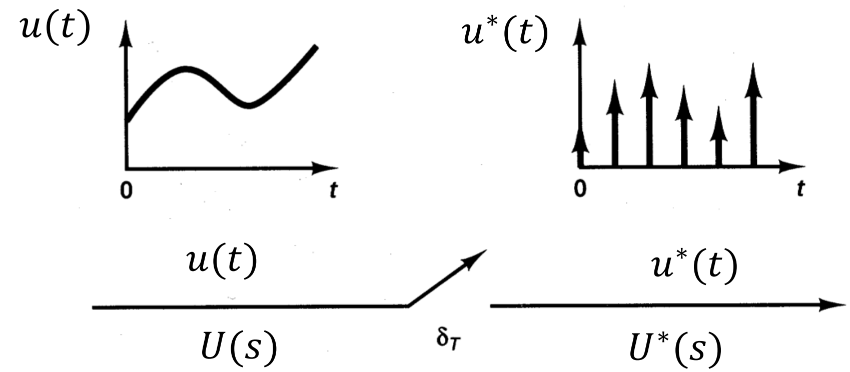
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Relation between Laplace Transform and Z-Transform

- Given the impulse train representation of a discrete-time signal

$$u^*(t) = u_o\delta(t) + u_1\delta(t-T) + u_2\delta(t-2T) + \cdots + u_k\delta(t-kT)$$

$$u^*(t) = \sum_{k=0}^{\infty} u_k\delta(t-kT)$$



- The Laplace Transform of above equation is

$$U^*(s) = u_o + u_1e^{-sT} + u_2e^{-2sT} + \cdots + u_ke^{-ksT}$$

$$U^*(s) = \sum_{k=0}^{\infty} u_ke^{-ksT}$$

- Let *z* be defined by

$$z^{-1} = e^{-sT}$$

$$U^*(s) = u_o + u_1 e^{-sT} + u_2 e^{-2sT} + \cdots + u_k e^{-ksT}$$

$$U^*(s) = \sum_{k=0}^{\infty} u_k e^{-ksT}$$

- Let z be defined by $z^{-1} = e^{-sT}$
- And $U(z) = U^*(s)$
- Given the causal sequence $\{u_0, u_1, u_2, \ldots, u_k\}$, its z-transform is defined as

$$U(z) = u_o + u_1 z^{-1} + u_2 z^{-2} + u_k z^{-k}$$

$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k}$$

- z^{-1} is a time delay operator.

Discrete-time state space representations

State space realizations

- State space realizations can be obtained from input-output time domain or z-domain models.
- Every system has infinite state space realizations.
- Standard canonical forms have special desirable properties.
- These realizations play an important role in digital filter design or controller implementation.

Controllable canonical realization

- Obtain the controllable canonical realization of the difference equation:

$$\dddot{y} + 3\ddot{y}(t) + 6\dot{y}(t) + 11y(t) = u(t)$$

- The discrete form of the system will be described by the following difference equation:

$$y(k+3) + 3y(k+2) + 6y(k+1) + 11y(k) = u(k)$$

- Number of states = order of the system

- $y(k+3) + 3y(k+2) + 6y(k+1) + 11y(k) = u(k)$
- Let

$$x_1(k) = y(k)$$
$$x_2(k) = y(k+1)$$
$$x_3(k) = y(k+2)$$

- So

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_3(k)$$
$$x_3(k+1) = u(k) - 11x_1(k) - 6x_2(k) - 3x_3(k)$$

- In discrete system: the state space form will be in the form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -11 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -11 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$$x\,(k+1) = A\,x\,(k) + B\,u(k)$$

$$y(k) = C\,x\,(k)$$

Controllable canonical realization

Obtain the controllable canonical realization of the Transfer Function:

$$\frac{y(z)}{u(z)} = \frac{3z + 3}{z^3 + 2z^2 + z + 3}$$

Note:

$$[z^3 + 2z^2 + z + 3\,]y(z) = [3z + 3\,]\,u(z)$$

$$z^3 y(z) \rightarrow y(k+3)$$
$$x\,(k) = [x_1(k)\;\; x_2(k)\;\;\; x_3(k)]^T$$
$$x\,(k) = [y(k)\;\; y(k+2)\;\;\; y(k+3)]^T$$

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z TRANSFORM METHOD FOR SOLVING DIFFERENCE EQUATIONS

z transform of $x(k+m)$ and $x(k-m)$

Discrete function	*z* Transform
$x(k+4)$	$z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - zx(3)$
$x(k+3)$	$z^3 X(z) - z^3 x(0) - z^2 x(1) - zx(2)$
$x(k+2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1} X(z)$
$x(k-2)$	$z^{-2} X(z)$
$x(k-3)$	$z^{-3} X(z)$
$x(k-4)$	$z^{-4} X(z)$

$$\frac{y(z)}{u(z)} = \frac{3z + 3}{z^3 + 2z^2 + z + 3}$$

Or simply

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} x(k)$$

<u>Observable form</u>

$$x(k+1) = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k)$$

Example

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2 + 1.3z + 0.4}$$

- **<u>CONTROLLABLE CANONICAL FORM:</u>**

$$x\,(k+1) = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix}\, x\,(k) \; + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix}\, x\,(k)$$

- **<u>OBSERVABLE CANONICAL FORM:</u>**

$$x\,(k+1) = \begin{bmatrix} -1.3 & 1 \\ -0.4 & 0 \end{bmatrix}\, x\,(k) \; + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\, x\,(k)$$

- **DIAGONAL CANONICAL FORM:**

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2 + 1.3z + 0.4}$$

- Using partial fraction technique

$$\frac{Y(z)}{U(z)} = \frac{5/3}{z+0.5} + \frac{-2/3}{z+0.8}$$

$$x\,(k+1) = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.8 \end{bmatrix} x\,(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 5/3 & -2/3 \end{bmatrix} x\,(k)$$

TF from state apace representation

$$x\,(k+1) = A\,x\,(k) + B\,u(k)$$
$$y(k) = C\,x\,(k)$$

$$T.F. = C(ZI - A)^{-1}B$$

End of Lec1