

# Computer Control of Dynamic Systems

Lecture-7

*Reduced Order Observer*

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# *Reduced Order Observer*

- *In previous lectures, we estimated  $x_1(k)$  [position] and  $x_2(k)$  [velocity].*
- *If an accurate measurement of a state is available, it is not reasonable to estimate it.*
- *We need to estimate only the remaining states.*
- *The resulting observer is called a reduced-order observer.*
- *If the measurements are relatively inaccurate, the full-order observer may yield better results.*

# *Reduced Order Observer*

- *We will partition the state vector as*

$$x(k) = \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix}$$

Where:  $x_a(k)$  : measurable states (known states)

$x_b(k)$  : unmeasurable states (unknown states to be estimated)

$$x(k+1) = A x(k) + B u(k)$$

$$y(k) = C x(k)$$

- *The plant state equation can be partitioned as*

$$\begin{bmatrix} x_a(k+1) \\ x_b(k+1) \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u(k)$$

$$y(k) = [I \quad 0] \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix}$$

$$\begin{bmatrix} x_a(k+1) \\ x_b(k+1) \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u(k)$$

- The equations for the measured states can be written as

$$x_a(k+1) = A_{aa} x_a(k) + A_{ab} x_b(k) + B_a u(k)$$

$$x_a(k+1) - A_{aa} x_a(k) - B_a u(k) = A_{ab} x_b(k)$$

- The equations for the estimated states are

$$x_b(k+1) = A_{ba} x_a(k) + A_{bb} x_b(k) + B_b u(k)$$

$$x_b(k+1) = A_{bb} x_b(k) + [A_{ba} x_a(k) + B_b u(k)]$$

- The term  $[A_{ba} x_a(k) + B_b u(k)]$  is then considered to be the "known inputs."

- Compare the state equations for the full-order observer to those for the reduced order observer.

$$x(k+1) = A x(k) + Bu(k)$$

$$x_b(k+1) = A_{bb} x_b(k) + A_{ba} x_a(k) + B_b u(k)$$

$$y(k) = C x(k)$$

$$x_a(k+1) - A_{aa} x_a(k) - B_a u(k) = A_{ab} x_b(k)$$

*By Analogy*

<i>Full order</i>	<i>Reduced order</i>
$X(k)$	$X_b(k)$
$A$	$A_{bb}$
$Bu(k)$	$A_{ba} x_a(k) + B_b u(k)$
$y(k)$	$X_a(k+1) - A_{aa} X_a(k) - B_a u(k)$
$C$	$A_{ab}$

- If we make these substitutions into the full-order observer equations

$$q(k + 1) = (A - GC)q(k) + Gy(k) + Bu(k)$$

- we obtain the equations

$$q_b(k+1) = (A_{bb} - GA_{ab}) q_b(k) + G [x_a(k+1) - A_{aa} x_a(k) - B_a u(k)] + A_{ba} x_a(k) + B_b u(k)$$

- Replacing  $x_a(k) \rightarrow Y(k)$  the equation can be written as

$$q_b(k+1) = (A_{bb} - G A_{ab}) q_b(k) + G Y(k+1) + [A_{ba} - G A_{aa}] Y(k) + [B_b - GB_a] u(k)$$

- The reduced order observer c/cs eq<sup>n</sup> is

$$\alpha_c (A) = |ZI - A_{bb} + G A_{ab}| = 0$$

- $G|_{\text{reduced order}}$  using Ackermann's formula:

$$G|_{\text{reduced order}} = \alpha_e (A_{bb}) \begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \\ A_{ab}A_{bb}^2 \\ \vdots \\ A_{ab}A_{bb}^{n-2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$



- Control law

$$\begin{aligned}
 u(k) &= -k \begin{bmatrix} x_a(k) \\ q_b(k) \end{bmatrix} = - [k_1 \quad k_b] \begin{bmatrix} x_a(k) \\ q_b(k) \end{bmatrix} \\
 &= -k_1 x_a(k) - k_b q_b(k)
 \end{aligned}$$

$$u(k+1) = -k_1 x_a(k+1) - k_b q_b(k+1)$$

- The transfer function  $D_{ce}(z)$  of the digital controller

$$D_{ce}(z) = \frac{-U(z)}{Y(z)} = K_1 + K_b [Z_I - A_{bb} + GA_{ab} + (B_b - GB_a)k_b]^{-1} \cdot [Gz + \{A_{ba} - GA_{aa} - k_1(B_b - G_{ba})\}]$$

*Example*

- Consider the design of the system

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k)$$

- The measuring state is  $X_1(k)$ , and we will estimate state is  $X_2(k)$ .
- The closed-loop system characteristic equation is

$$\alpha_c(Z) = Z^2 - 1.776Z + 0.819 = 0$$

- As in previous Example chose the estimator characteristic-equation roots to be at  $z = 0.819$ .
- The reduced-order observer is first order is

$$\alpha_e(Z) = Z - 0.819$$

- From the plant state equations, the partitioned matrices are seen to be

$$\begin{aligned}
 A_{aa} &= 1 & A_{ab} &= 0.0952 \\
 A_{ba} &= 0 & A_{bb} &= 0.905 \\
 B_a &= 0.00484 & B_b &= 0.0952
 \end{aligned}$$

$$G = \alpha_e (A_{bb}) \left[ A_{ab} \right]^{-1} [1]$$

$$\alpha_e (A_{bb}) = A_{bb} - 0.819I = 0.905 - 0.819 = 0.086$$

$$G = 0.086 * (0.0952)^{-1} * 1 = 0.903$$

- The reduced order observer equation is

$$q_b(k+1) = (A_{bb} - GA_{ab}) q_b(k) + G Y(k+1) + [A_{ba} - GA_{aa}] Y(k) + [B_b - GB_a] u(k)$$

$$= 0.819 q_b(k) + 0.903 Y(k+1) - 0.903 Y(k) + 0.0908 u(k)$$

- Here  $q_b(k)$  is the estimate of the state  $x_2(k)$ .
- $q_b(k) = 0.819 q_b(k-1) + 0.903 y(k) - 0.903 y(k-1) + 0.0908 u(k-1)$
- $q(k)$  is the estimate at the present time

- From a previous example the control law is given by

$$u(k) = -4.52 y(k) - 1.12 x_2(k)$$

which is implemented as

$$u(k) = -4.52 x_1(k) - 1.12 q_b(k)$$

Hence we can write the observer equation as

$$q_b(k+1) = 0.819q_b(k) + 0.903y(k+1) - 0.903y(k) + 0.0908[-4.52y(k) - 1.12q_b(k)]$$

Or

$$q_b(k+1) = 0.717 q_b(k) + 0.903 y(k+1) - 1.313 y(k)$$

and

$$q_b(k) = 0.717 q_b(k-1) + 0.903 y(k) - 1.313 y(k-1)$$

$$D_{ce}(z) = \frac{-U(z)}{Y(z)} = K_1 + K_b [Z_I - A_{bb} + GA_{ab} + (B_b - G_{ba})k_b]^{-1} \cdot [Gz + \{A_{ba} - GA_{aa} - k_1(B_b - G_{ba})\}]$$

$$= 4.52 + 1.12[z - 0.905 + (0.903)(0.0952) + \{0.0952 - (0.903)(0.00484)\}1.12]^{-1} \cdot \{0.903z + [0 - (0.903)(1) - 4.52\{0.0952 - (0.903)(0.00484)\}]\}$$

$$= 4.52 + \frac{1.12(0.903z - 1.314)}{z - 0.717} = \frac{5.53z - 4.71}{z - 0.717} = \frac{5.53(z - 0.852)}{z - 0.717}$$

*End Of Lecture*