Computer Control of Dynamic Systems

Lecture-7

Reduced Order Observer

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Reduced Order Observer

- *In previous lectures, we estimated $x_1(k)$ [position] and $x_2(k)$ [velocity].*
- *If an accurate measurement of a state is available, it is not reasonable to estimate it.*
- *We need to estimate only the remaining states.*
- *The resulting observer is called a **reduced-order observer**.*
- *If the measurements are relatively inaccurate, the full-order observer may yield better results.*

Reduced Order Observer

- *We will partition the state vector as*

$$x(k) = \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix}$$

Where: $x_a(k)$: measurable states (known states)

$x_b(k)$: unmeasurable states (unknown states to be estimated)

$$x(k+1) = A\,x(k) + Bu(k)$$

$$y(k) = C\,x(k)$$

- *The plant state equation can be partitioned as*

$$\begin{bmatrix} x_a(k+1) \\ x_b(k+1) \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u(k)$$

$$y(k) = [I \quad 0] \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix}$$

$$\begin{bmatrix} x_a(k+1) \\ x_b(k+1) \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} \mathrm{u(k)}$$

- The equations for the measured states can be written as

$$x_a\,(k+1) = A_{aa}\,x_a(k) + A_{ab}\,x_b(k) + B_a\,u(k)$$

$$x_a\,(k+1) - A_{aa}\,x_a(k) - B_a\,u(k) = A_{ab}\,x_b(k)$$

- The equations for the estimated states are

$$x_b\,(k+1) = A_{ba}\,x_a(k) + A_{bb}\,x_b(k) + B_b\,u(k)$$

$$x_b\,(k+1) = A_{bb}\,x_b(k) + [A_{ba}\,x_a(k) + B_b\,u(k)]$$

- The term $[A_{ba}\ x_a(k) + B_b\ u(k)]$ is then considered to be the "known inputs."

- *Compare the state equations for the full-order observer to those for the reduced order observer.*

$$x(k+1) = A\,x(k) + Bu(k)$$
$$x_b\,(k+1) = A_{bb}\,x_b(k) + A_{ba}\,x_a(k) + B_b\,u(k)$$

$$y(k) = C\,x(k)$$
$$x_a\,(k+1) - A_{aa}\,x_a(k) - B_a\,u(k) = A_{ab}\,x_b(k)$$

By Analogy

Full order	***Reduced order***
$X(k)$	$X_b(k)$
A	A_{bb}
$Bu(k)$	$A_{ba}\,x_a(k) + B_b\,u(k)$
$y(k)$	$X_a\,(k+1) - A_{aa}\,X_a(k) - B_a\,u(k)$
C	A_{ab}

- If we make these substitutions into the full-order observer equations

$$q(k + 1) = (A - GC)q(k) + Gy(k) + Bu(k)$$

- we obtain the equations

$$q_b (k+1) = (A_{bb} – GA_{ab}) q_b(k) + G [x_a (k+1) - A_{aa} x_a(k) - B_a u(k)] + A_{ba} x_a(k) + B_b u(k)$$

- Replacing $x_a(k) \rightarrow Y(k)$ the equation can be written as

$$q_b (k+1) = (A_{bb} – G A_{ab}) q_b(k) + G Y(k+1) +[A_{ba} – G A_{aa}]Y(k) +[B_b -GB_a] u(k)$$

- The reduced order observer c/cs eq$^{\underline{n}}$ is

$$\propto_c (A) = |ZI - A_{bb} + G\,A_{ab}| = 0$$

- $G|_{\text{reduced order}}$ using Ackermann's formula:

$$G\,|_{\text{reduced order}} = = \propto_e (A_{bb}) \begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \\ A_{ab}A_{bb^2} \\ . \\ . \\ A_{ab}A_{bb^{n-2}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 1 \end{bmatrix}$$

- Control low

$$u(k) = \text{-}k \begin{bmatrix} x_a(k) \\ q_b(k) \end{bmatrix} = \text{-} \begin{bmatrix} k_1 & k_b \end{bmatrix} \begin{bmatrix} x_a(k) \\ q_b(k) \end{bmatrix}$$

$$= \text{-}\ k_1\ x_a\ (k) - k_b\ q_b(k)$$

$$u(k+1) = \text{-}k_1\ x_a\ (k+1) - k_b\ q_b(k+1)$$

- The transfer function $D_{ce}(z)$of the digital controller

$$D_{ce}\ (z) = \frac{-U(z)}{Y(z)} = K_1 + K_b\ [Z_I - A_{bb} + GA_{ab} + (B_b\ \text{-}\ GBa)k_b]^{-1}\ .\ [Gz + \{A_{ba} - GA_{aa} - k_1(B_b - G_{ba})\}]$$

Example

- Consider the design of the system

$$x(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} x(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k)$$

- The measuring state is X_1 (k), and we will estimate state is X_2(k).

- The closed-loop system characteristic equation is

$$\propto_c (Z) = Z^2 - 1.776Z + 0.819 = 0$$

- As in previous Example chose the estimator characteristic-equation roots to be at $z = 0.819$.

- The reduced-order observer is first order is

$$\propto_e (Z) = Z - 0.819$$

- From the plant state equations, the partitioned matrices are seen to be

$$\mathrm{A_{aa}} = 1 \qquad \mathrm{A_{ab}} = 0.0952$$
$$\mathrm{A_{ba}} = 0 \qquad \mathrm{A_{bb}} = 0.905$$
$$\mathrm{B_a} = 0.00484 \quad \mathrm{B_b} = 0.0952$$

$$\mathrm{G} = \propto_e (\mathrm{A_{bb}}) \begin{bmatrix} A_{ab} \\ \\ \end{bmatrix}^{-1} [1]$$

$$\propto_e (\mathrm{A_{bb}}) = \mathrm{A_{bb}} - 0.819\mathrm{I} = 0.905 - 0.819 = 0.086$$

$$\mathrm{G} = 0.086 * (0.0952)^{-1} * 1 = 0.903$$

- The reduced order observer equation is

$$q_b(k+1) = (A_{bb} - GA_{ab})\, q_b(k) + G\, Y(k+1) + [\, A_{ba} - GA_{aa}\,]Y(k) + [B_b - GB_a]\, u(k)$$

$$= 0.819\, q_b\,(k) + 0.903\, Y(k+1) - 0.903\, Y(k) + 0.0908\, u(k)$$

- Here $q_b\,(k)$ is the estimate of the state $x_2(k)$.

- $q_b\,(k) = 0.819\, q_b\,(k - 1) + 0.903\, y(k) - 0.903\, y(k - 1) + 0.0908\, u(k - 1)$

- $q(k)$ is the estimate at the present time

- From a previous example the control law is given by

$$u(k) = -4.52\, y(k) - 1.12\, x_2(k)$$

which is implemented as

$$u(k) = -4.52\, x_1(k) - 1.12\, q_b(k)$$

Hence we can write the observer equation as

$$q_b(k+1) = 0.819 q_b(k) + 0.903 y(k+1) - 0.903 y(k) + 0.0908[\, -4.52 y(k) - 1.12 q_b(k)]$$

Or

$$q_b(k+1) = 0.717\, q_b(k) + 0.903\, y(k+1) - 1.313\, y(k)$$

and

$$q_b(k) = 0.717\, q_b(k-1) + 0.903\, y(k) - 1.313\, y(k-1)$$

$$D_{ce}\ (z) = \frac{-U(z)}{Y(z)} = K_1 + K_b\ [Z_I - A_{bb} + GA_{ab} + (B_b - G\ Ba)k_b]^{-1} .\ [Gz + \{A_{ba} - GA_{aa} - k_1(B_b - G_{ba})\}]$$

$$= 4.52 + 1.12[z - 0.905 + (0.903)(0.0952) + \{0.0952 - (0.903)(0.00484)\}1.12]^{-1}$$

$$.\ \{0.903\ z + [0 - (0.903)(1) - 4.52\{0.0952 - (0.903)(0.00484\})]\}$$

$$= 4\ 52 + \frac{1.12(0.903z - 1.314)}{z - 0.717} = \frac{5.53\ z - 4.71}{z - 0.717} =$$

$$= \frac{5.53(z - 0.852)}{z - 0.717}$$

End Of Lecture