Finite Element Method

Lecture 10

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Outline

- Finite Element Method
- Direct Method
- 1D Example

Reference

- S. Chapra and R. Canale, “Numerical Method’s for Engineers”, McGraw-Hill, 5th Ed., 2006

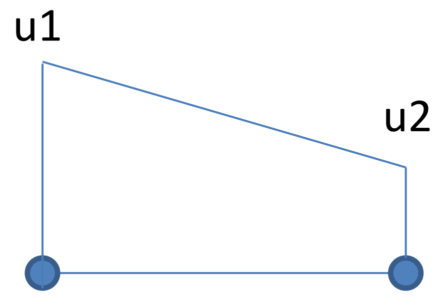
Finite Element Method

- Preferred for structures with irregular geometries:
  - Finite Difference Method suffers from approximations at irregular boundaries
- Divides the structure into elements:
  1. Apply governing equations within elements, with possible approximations or numerical interpolations
  2. Apply continuity at boundaries of these elements
  3. Assemble these elements in one matrix

Element Equations

Aproximate the solution within the element, e.g. straight line:

$$u(x) = a_0 + a_1 x$$



For a single element between nodes 1 and 2:

$$u(x_1) = u_1 = a_0 + a_1 x_1$$
$$u(x_2) = u_2 = a_0 + a_1 x_2$$

$$a_0 = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1}, \qquad a_1 = \frac{u_2 - u_1}{x_2 - x_1}$$

$$u = N_1 u_1 + N_2 u_2, \qquad N_1 = \frac{x_2 - x}{x_2 - x_1}, \qquad N_2 = \frac{x - x_1}{x_2 - x_1}$$

N_1 and N_2 are *Linear Interpolation Functions*

Element Equations (2)

$$\frac{du}{dx} = \frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2$$

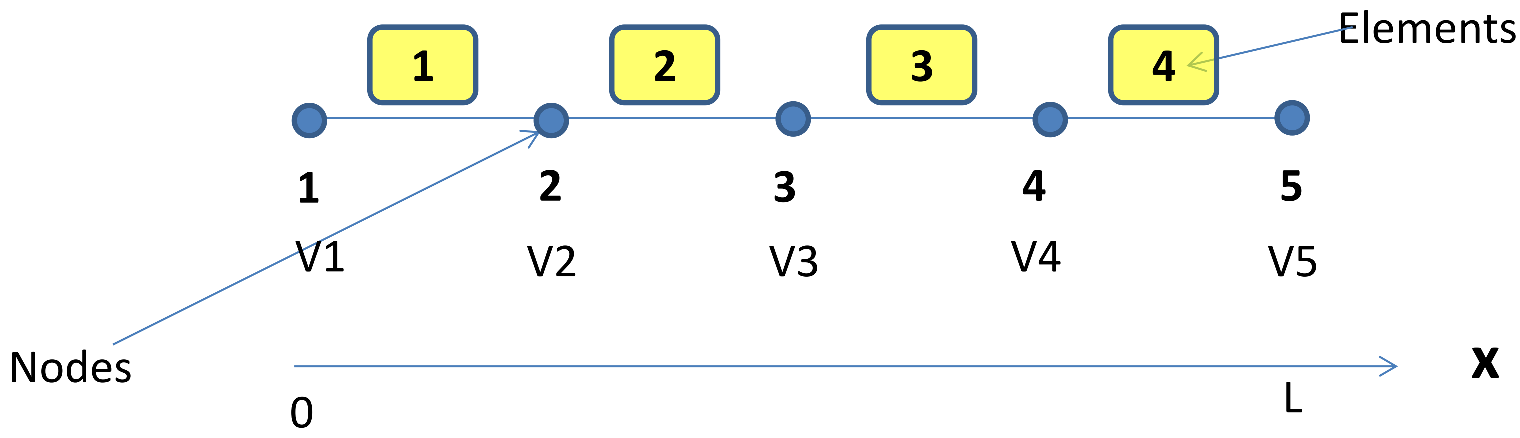
$$\frac{dN_1}{dx} = \frac{-1}{x_2 - x_1}, \qquad \frac{dN_2}{dx} = \frac{1}{x_2 - x_1}$$

$$\frac{du}{dx} = \frac{1}{x_2 - x_1}(-u_1 + u_2)$$ → Straight Line slope

$$\int_{x1}^{x2} u\, dx = \int_{x1}^{x2} [N_1u_1 + N_2u_2\,]\, dx = \frac{u_1 + u_2}{2}(x_2 - x_1)$$

Since $\int_{x1}^{x2} Nu\, dx = \frac{1}{2}(x_2 - x_1)$

1D Example:
Poisson's Equation for Potential



$$\frac{d^2V(x)}{dx^2} = -f(x)$$

where $f(x) = \rho/\epsilon$

Boundary Conditions: $V(0) = V_1, \quad V(L) = V_5$

The Direct Method

- For 1st Order D.E:

Example: Drift Current Density *in an element* (A/m²):

$$J = \sigma E = -k\frac{dV}{dx}$$

- For $f(x) = 0$ in 2nd order D.E. :

Example: Poisson's equation in dielectric with no charges (capacitor):

$$\frac{d^2V(x)}{dx^2} = 0 \rightarrow \frac{dV}{dx} = k_2 = -D/\epsilon$$

Direct Method Example: **1D Drift Current**

At the boundary of each elements J is continuous

$$j_1 = k\frac{V_1 - V_2}{x_2 - x_1}, \qquad j_2 = k\frac{V_2 - V_1}{x_2 - x_1}$$

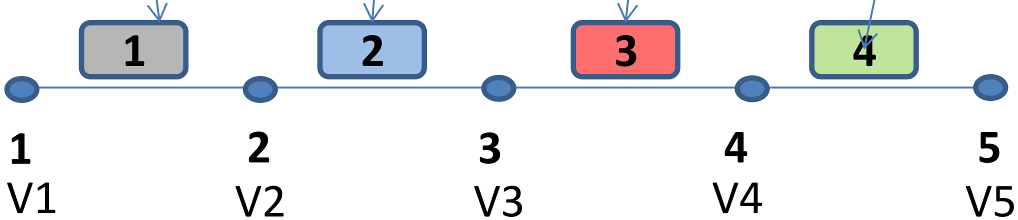
Local Matrix Equations:

$$\frac{1}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\frac{dV(x_1)}{dx} \\ \frac{dV(x_2)}{dx} \end{bmatrix}$$

Assembly of Local Equations into One Matrix

Boundary Conditions: $V(0) = 40V,\ \ V(L) = 200V,\ x_2 - x_1 = 2.5cm$

$$\begin{bmatrix} 0.4 & -0.4 & & & \\ -0.4 & 0.4 + 0.4 & -0.4 & & \\ & -0.4 & 0.4 + 0.4 & -0.4 & \\ & & -0.4 & 0.4 + 0.4 & -0.4 \\ & & & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} V1 \\ V2 \\ V3 \\ V4 \\ V5 \end{bmatrix} = \begin{bmatrix} -\dfrac{dV(x_1)}{dx} \\ \dfrac{dV(x_2)}{dx} - \dfrac{dV(x_2)}{dx} \\ \dfrac{dV(x_3)}{dx} - \dfrac{dV(x_3)}{dx} \\ \dfrac{dV(x_4)}{dx} - \dfrac{dV(x_4)}{dx} \\ \dfrac{dV(x_5)}{dx} \end{bmatrix}$$



Boundary Conditions for Elements

Solve Matrix Using BCs for Structure

- Now, remove $V_1 = 40V$ and $V_5 = 200V$ from matrix (BC):

$$\begin{bmatrix} 0.4 & -0.4 & & & \\ -0.4 & 0.8 & -0.4 & & \\ & -0.4 & 0.8 & -0.4 & \\ & & -0.4 & 0.8 & -0.4 \\ & & & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 40 \\ V2 \\ V3 \\ V4 \\ 200 \end{bmatrix} = \begin{bmatrix} -\frac{dV(x_1)}{dx} \\ 0 \\ 0 \\ 0 \\ \frac{dV(x_5)}{dx} \end{bmatrix}$$

- Substitute for V1 & V5 and solve a **3x3** matrix (LU, GE, …):

$$\begin{bmatrix} 0.8 & -0.4 & \\ -0.4 & 0.8 & -0.4 \\ & -0.4 & 0.8 \end{bmatrix} \begin{bmatrix} V2 \\ V3 \\ V4 \end{bmatrix} = \begin{bmatrix} (0.4)(40) \\ 0 \\ (0.4)(200) \end{bmatrix}$$

- Find V2,V3,V4: $V_2 = 80V,\ V_3 = 120V, V_4 = 160V$

Solve Matrix and find BCs for Structure

- First Equation: $0.4\,V_1 - 0.4\,V_2 = -\frac{dV(x_1)}{dx}$

$$\frac{dV(x_1)}{dx} = -(0.4)(40) + (0.4)(80) = 16\,V/cm$$

- Similarly for last equation and V_5=200V:

$$-0.4\,V_4 + 0.4\,V_5 = \frac{dV(x_5)}{dx}$$

$$\frac{dV(x_5)}{dx} = -(0.4\,)(160) + (0.4)(200) = 16\,V/cm$$