Finite Element Method (2)

Lecture 11

Dr. Amr Bayoumi

Fall 2014

Advanced Engineering Mathematics (EC760)

Arab Academy for Science and Technology - Cairo

Outline

- Order Reduction using Integration By Parts
- Galerkin’s Method
- 2D using Triangular Elements
- 2D Example

Reference

- S. Chapra and R. Canale, "Numerical Method's for Engineers", McGraw-Hill, 5^{th} Ed., 2006

Order Reduction

- 2nd order differential equations can be reduced to 1st order:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\int_{x1}^{x2} \frac{d^2y}{dx^2}dx = \int_{x1}^{x2} \frac{d}{dx}\left[\frac{dy}{dx}\right]dx = \int_{x1}^{x2} d\left[\frac{dy}{dx}\right]$$

Interpolation Methods

- Use interpolation functions, and try to minimize residual errors

Residual Minimization

$$\frac{d^2V(x)}{dx^2} = -f(x) \qquad \rightarrow \qquad \frac{d^2V(x)}{dx^2} + f(x) = 0$$

This can be expressed as a "*Residual*" equation:

$$R = \frac{d^2V(x)}{dx^2} + f(x)$$

R needs to be minimized towards zero, using the *Method of Weighted Residuals (MWR)*:

$$\int_D R\, W_i dD = 0, \qquad i = 1, 2, \ldots m$$

N_i = linearly independent weighting functions

D = solution domain

i = indices for domain (eg. x_1 to x_2, y_1 to y_2, …)

Galerkin's Method

- Use N_i as the weighting function W_i.
- Example: 1D element between x_1 and x_2. Then the domain D is x, ranging between $x_1 \rightarrow x_2$ $(i = 1,2)$:

$$\int_{x1}^{x2} \left[\frac{d^2V}{dx^2} + f(x) \right] N_i \, dx = 0 \,, \qquad i = 1,2$$

$$\int_{x1}^{x2} \left[\frac{d^2V}{dx^2} \right] N_i dx = -\int_{x1}^{x2} [f(x)] \, N_i dx$$

Galerkin's Method (2)

Using integration by parts:

$$u = N_i, \quad du = dN_i = \left[\frac{d\,N_i}{dx}\right]dx, \quad dv = \left[\frac{d^2V}{dx^2}\right]dx = d\left[\frac{dV}{dx}\right]$$

$$\int_{x1}^{x2} N_i \left[\frac{d^2V(x)}{dx^2}\right] dx = N_i \frac{dV}{dx}\Big|_{x_1}^{x_2} - \int_{x1}^{x2} \frac{dV}{dx}\frac{dN_i}{dx} dx$$

For i=1, $N_1(x_2)=0$, $N_1(x_1)=1$: $\quad N_1\frac{dV}{dx}\Big|_{x_1}^{x_2} = -\frac{dV(x_1)}{dx}$

For i=2, $N_2(x_2)=1$, $N_2(x_1)=0$: $\quad N_2\frac{dV}{dx}\Big|_{x_1}^{x_2} = \frac{dV(x_2)}{dx}$

Galerkin's Method (3)

$$\int_{x1}^{x2} \left[\frac{d^2V}{dx^2}\right] N_i dx = -\int_{x1}^{x2} [f(x)]\, N_i dx$$

For i=1, 2:

$$\int_{x1}^{x2} \frac{dV}{dx}\frac{dN_1}{dx} dx = -\frac{dV(x_1)}{dx} + \int_{x1}^{x2} [f(x)]\, N_1 dx$$

$$\int_{x1}^{x2} \frac{dV}{dx}\frac{dN_2}{dx} dx = \frac{dV(x_2)}{dx} + \int_{x1}^{x2} [f(x)]\, N_2 dx$$

From previous lecture for dNi/dx:

$$\int_{x1}^{x2} \frac{dV}{dx}\frac{dN_1}{dx} dx = \int_{x1}^{x2} \frac{V_2 - V_1}{x_2 - x_1}\left[\frac{-1}{x_2 - x_1}\right] dx = \frac{1}{x_2 - x_1}(V_1 - V_2)$$

$$\int_{x1}^{x2} \frac{dV}{dx}\frac{dN_2}{dx} dx = \int_{x1}^{x2} \frac{V_2 - V_1}{x_2 - x_1}\left[\frac{1}{x_2 - x_1}\right] dx = \frac{1}{x_2 - x_1}(-V_1 + V_2)$$

Galerkin's Method (4): **Local Matrix**

$$\frac{1}{x_2 - x_1}(V_1 - V_2) = -\frac{dV(x_1)}{dx} + \int_{x1}^{x2} [f(x)]\, N_1 dx$$

$$\frac{1}{x_2 - x_1}(-V_1 + V_2) = \frac{dV(x_2)}{dx} + \int_{x1}^{x2} [f(x)]\, N_2 dx$$

In Matrix form, all D.E. terms are reduced to 0th and 1st order:

$$\frac{1}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{dV(x_1)}{dx} \\ \dfrac{dV(x_2)}{dx} \end{bmatrix} + \begin{bmatrix} \int_{x1}^{x2} [f(x)]\, N_1 dx \\ \int_{x1}^{x2} [f(x)]\, N_2 dx \end{bmatrix}$$

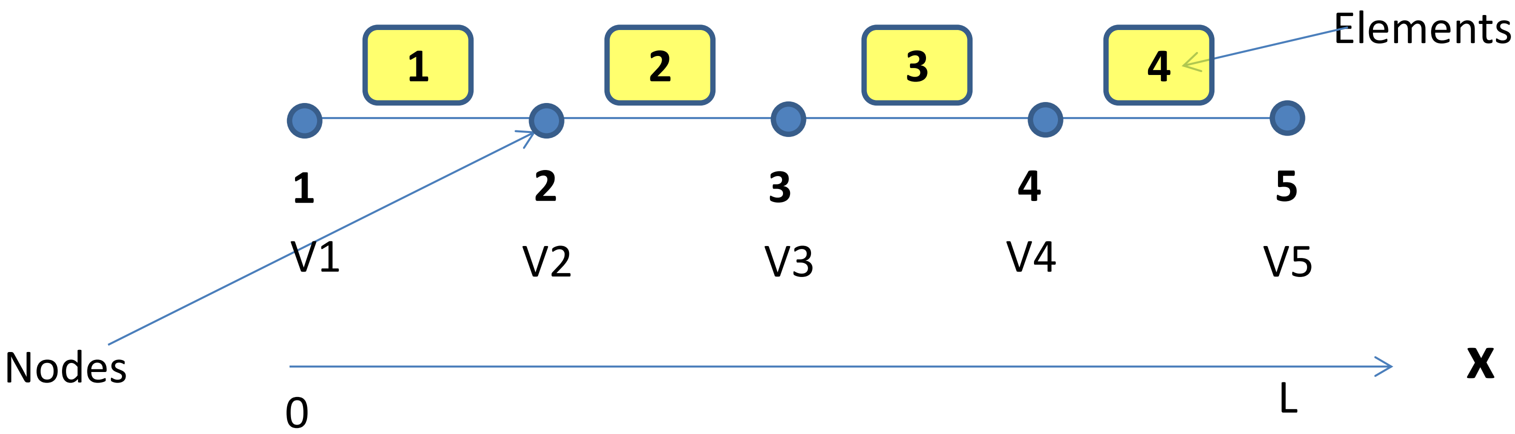
Nodes

Element Stiffness Matrix

Boundary Conditions

External Effects

1D Example:
Poisson's Equation for Potential $(f(x) \neq 0)$



$$\frac{d^2V(x)}{dx^2} = -f(x)$$

where $f(x) = \rho/\epsilon$

Boundary Conditions: $V(0) = V_1, \quad V(L) = V_5$

Assembly of Local Equations into Global Matrix

- $f(x) = 10V/cm^2$
- Boundary Conditions:

$$V(0) = 40V, \qquad V(L) = 200V, \qquad x_2 - x_1 = 2.5cm$$

$$[K][V_i] = [BC] + [E]$$

Stiffness Matrix

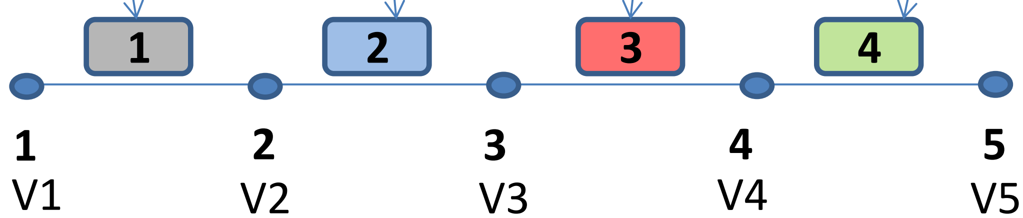
Nodes

Boundary Conditions

External Effects

Global Stiffness Matrix

$$[K] = \begin{bmatrix} 0.4 & -0.4 & & & \\ -0.4 & 0.4 + 0.4 & -0.4 & & \\ & -0.4 & 0.4 + 0.4 & -0.4 & \\ & & -0.4 & 0.4 + 0.4 & -0.4 \\ & & & -0.4 & 0.4 \end{bmatrix}$$



Boundary Conditions for Global Matrix

$$[BC]=\begin{bmatrix} -\dfrac{dV(x_1)}{dx} \\ \dfrac{dV(x_2)}{dx}-\dfrac{dV(x_2)}{dx} \\ \dfrac{dV(x_3)}{dx}-\dfrac{dV(x_3)}{dx} \\ \dfrac{dV(x_4)}{dx}-\dfrac{dV(x_4)}{dx} \\ \dfrac{dV(x_5)}{dx} \end{bmatrix}=\begin{bmatrix} -\dfrac{dV(x_1)}{dx} \\ 0 \\ 0 \\ 0 \\ \dfrac{dV(x_5)}{dx} \end{bmatrix}$$

External Effects for Global Matrix

$$[E] = \begin{bmatrix} \int_{x1}^{x2} [f(x)]\, N_1 dx \\ \int_{x1}^{x2} [f(x)]\, N_2 dx + \int_{x2}^{x3} [f(x)]\, N_2 dx \\ \int_{x2}^{x3} [f(x)]\, N_3 dx + \int_{x3}^{x4} [f(x)]\, N_3 dx \\ \int_{x3}^{x4} [f(x)]\, N_4 dx + \int_{x4}^{x5} [f(x)]\, N_4 dx \\ \int_{x4}^{x5} [f(x)]\, N_5 dx \end{bmatrix} = \begin{bmatrix} 12.5 \\ 12.5 + 12.5 \\ 12.5 + 12.5 \\ 12.5 + 12.5 \\ 12.5 \end{bmatrix}$$

$f(x) = F_1 = 10V/cm^2,\ N_1 = \frac{x_2 - x}{x_2 - x_1},\ N_2 = \frac{x - x_1}{x_2 - x_1},$

$$F_1 \int_{x1}^{x2} \frac{x_2 - x}{x_2 - x_1} dx = \frac{F_1}{x_2 - x_1}\left[x_2 x - \frac{x^2}{2}\right]\Big|_{x_1}^{x_2} = \frac{F_1}{x_2 - x_1}\left\{\left[{x_2}^2 - \frac{{x_2}^2}{2}\right] - \left[x_2 x_1 - \frac{{x_1}^2}{2}\right]\right\}$$

$$= \frac{F_1/2}{x_2 - x_1}[x_2 - x_1]^2 = \frac{F_1}{2}(x_2 - x_1) = (\frac{10}{2}\ V/cm^2)(2.5cm) = 12.5V/cm$$

Solve Matrix Using BCs for Structure

- As before, remove $V_1 = 40V$ and $V_5 = 200V$ from matrix (BC):

$$\begin{bmatrix} 0.4 & -0.4 & & & \\ -0.4 & 0.8 & -0.4 & & \\ & -0.4 & 0.8 & -0.4 & \\ & & -0.4 & 0.8 & -0.4 \\ & & & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 40 \\ V2 \\ V3 \\ V4 \\ 200 \end{bmatrix} = \begin{bmatrix} -\dfrac{dV(x_1)}{dx} + 12.5 \\ 25 \\ 25 \\ 25 \\ \dfrac{dV(x_5)}{dx} + 12.5 \end{bmatrix}$$

- As in previous lecture, substitute for V1 & V5 and solve a **3x3** matrix (LU, GE, …):
- Find V_2, V_3, V_4: $V_2 = 173.75V,\ V_3 = 245V, V_4 = 253V$
- Solve for $\dfrac{dV(x_1)}{dx} = 66V/cm, \dfrac{dV(x_5)}{dx} = -34V/cm$