



Boundary Value and Initial Condition Problems in Partial Differential Equations: Wave Equation

Lecture 5

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Outline

- Wave Equation
 - Boundary Values
 - Initial Condition
- Separation of Variables
- Fourier Series Analysis

Draft



Wave Equation (PDE)

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

Example: Electromagnetic Waves:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \vec{E}(x, t) = 0 \quad \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \vec{B}(x, t) = 0$$

Boundary Conditions ($u(x, t)$ at certain x 's, at any t):

$$u(0, t) = 0, \quad u(L, t) = 0$$

Initial Conditions ($u(x, t)$ or its derivatives at $t=0$, at any x):

$$u(x, 0) = f(x) \quad \frac{\partial u(x, 0)}{\partial t} = g(x)$$

Examples: Vibrating string with fixed ends, EM standing waves



Separation of Variables

Let: $u(x, t) = F(x) G(t)$

$$\frac{\partial^2 u}{\partial t^2} = F \ddot{G} \quad , \quad \frac{\partial^2 u}{\partial x^2} = G F''$$

Substitute in Wave PDE:

$$F \ddot{G} = c^2 G F'' \quad \rightarrow \quad \frac{F''}{F} = \frac{\ddot{G}}{c^2 G} = k = \text{constant}$$

$k = \text{constant}$ since F''/F does not depend on t , and \ddot{G}/G does not depend on x , thus $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ of RHS & LHS is zero

$$F'' - k F = 0 \quad , \quad \ddot{G} - c^2 k G = 0$$



Boundary Conditions:

$$F'' - kF = 0$$

For the Differential Eqn: $F'' = kF$

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$

Case $k=0$:

$$F(x) = C_1 + C_2$$

Using BC:

$$F(0) = 0 = C_1 + C_2 = F(x) \rightarrow F(x)G(t) = 0 \text{ (Unacceptable)}$$

Case $k = +ve = \mu^2$:

$$F(0) = 0 = C_1 + C_2 \rightarrow F(x) = C_1(e^{\mu x} - e^{-\mu x}) = \sinh(\mu x)$$

$$F(L) = 0 \rightarrow \sinh(\mu L) \neq 0$$

Since $\sinh(\theta) = 0$ **ONLY** at $\theta=0$ \rightarrow k cannot be positive

$$\text{Therefore: } k = -ve = -p^2$$



Boundary Conditions:

$$F'' - kF = 0$$

$$k = -ve = -p^2: F(x) = C_1 e^{jpx} + C_2 e^{-jpx}$$

$$\text{Using BC: } F(0) = 0 = C_1 + C_2$$

$$\text{Using: } e^{j\theta} = \cos \theta + j \sin \theta \rightarrow F(x) = 2jC_1 \sin(px)$$

$$\text{Normalizing } 1 = 2jC_1 \rightarrow F(x) = \sin(px)$$

$$\text{Using BC: } F(L) = \sin(pL) = 0 \rightarrow pL = \pm n\pi \quad (n = 1, 2, 3, \dots)$$

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

Using Superposition of independent solutions of ODEs:

$$F(x) = \sum_{n=1}^{\infty} F_n(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)$$



Initial Conditions & Fourier Series in PDE:

$$\ddot{G} - c^2 k G = 0$$

ODE for any n : $\ddot{G}_n + c^2 p_n^2 G_n = 0 \rightarrow \ddot{G}_n + \lambda_n^2 G_n = 0$

$$\lambda_n = c p_n = c \frac{n\pi}{L}$$

General Solution for $G_n(t)$: $G_n(t) = A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t)$

Overall Soln.:

$$u(x, t) = \sum_{n=1}^{\infty} F_n(x) G_n(t) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t)] \sin\left(\frac{n\pi}{L} x\right)$$

Initial Conditions ($t=0$):

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n 0) + B_n \sin(\lambda_n 0)] \sin\left(\frac{n\pi}{L} x\right)$$
$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right) = f(x)$$

Using Fourier Series: $A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$



Initial Conditions & Fourier Series in PDE (2):

$$\ddot{G} - c^2 k G = 0$$

Overall Soln.:

$$u(x, t) = \sum_{n=1}^{\infty} F_n(x) G_n(t) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t)] \sin\left(\frac{n\pi}{L} x\right)$$

Initial Conditions ($t=0$): $\frac{\partial u(x, 0)}{\partial t} = g(x)$

$$\sum_{n=1}^{\infty} [-A_n \lambda_n \sin(\lambda_n t) + B_n \lambda_n \cos(\lambda_n t)] \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

$$\sum_{n=1}^{\infty} B_n \lambda_n \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

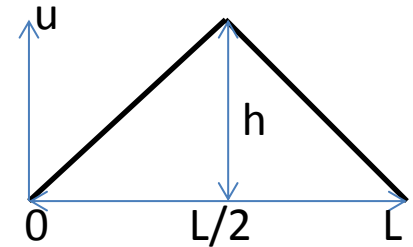
Using Fourier Series: $B_n \lambda_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$



Example: Triangular Initial Condition $u(x,0)$

$$u(x,0) = f(x) = \begin{cases} h \frac{2x}{a} & \text{for } 0 < x < \frac{L}{2} \\ h \left(2 - \frac{2x}{L}\right) & \text{for } \frac{L}{2} < x < L \end{cases}$$

$$\frac{\partial u(x,0)}{\partial t} = g(x) = 0 \rightarrow B_n = 0$$



$$A_n = \frac{2}{L} \int_0^{L/2} h \frac{2x}{a} \sin\left(\frac{n\pi}{L}x\right) dx + \frac{2}{L} \int_{L/2}^L h \left(2 - \frac{2x}{L}\right) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$A_n = \frac{8h}{\pi^2} \left[\sin\left(\frac{n\pi}{2}\right) / n^2 \right]$$

Using: $\sin(A) \cos(B) = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{n\pi}{L}(x-ct)\right) + A_n \sin\left(\frac{n\pi}{L}(x+ct)\right) \right]$$

Two travelling waves in opposite directions: Resulting in Standing Waves