Boundary Value Problems in Partial Differential Equations:

2-D Laplace Equation

(Potential Equation)

Lecture 6

Dr. Amr Bayoumi

Fall 2014

Advanced Engineering Mathematics (EC760)

Arab Academy for Science and Technology - Cairo

Outline

- 2D Laplace Equation
- Boundary Conditions:
  - Dirichlet BC
  - Neumann BC
  - Mixed BC
- Separation of Variables
- Fourier Series Analysis

References

- D.L. Powers, “Boundary Value Problems and Partial Differential Equations”, Academic Press, 6^{th} Ed., 2010

2D Laplace Equation (PDE)

$$\nabla^2 u(x,y) = \frac{\partial^2 u\,(x,y)}{\partial x^2} + \frac{\partial^2 u\,(x,y)}{\partial y^2} = 0$$

Example: Electrostatic potential in 2D

Poisson's Equation:

$$\nabla^2 u(x,y) = -H(x,y)$$

Boundary Conditions

- Dirichlet's BC:

  $u(x, y)$ predefined along boundaries

- Neumann's BC:

  $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ predefined along boundaries

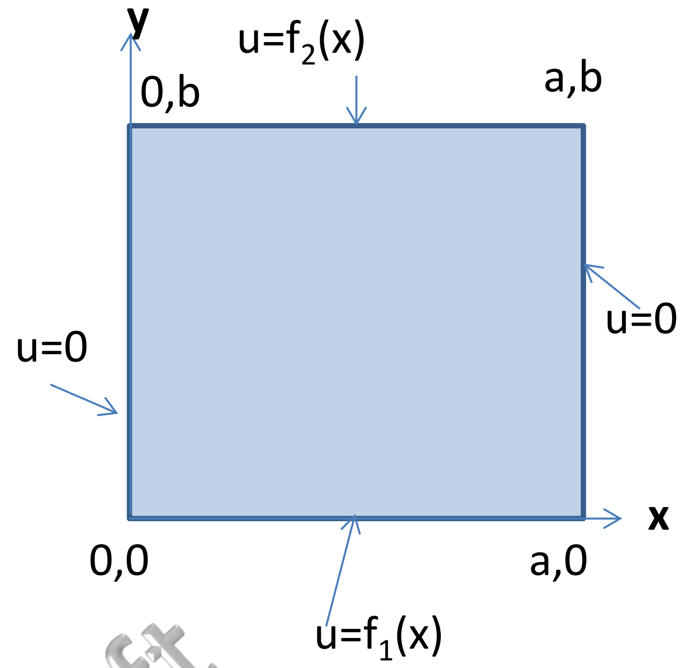
- Mixed BCs:

  Mix of $u(x, y)$ and $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial y}$ predefined along boundaries

Example Boundary Conditions: 2D Potential in a Rectangle

- Dirichlet's BC:

$$u(x,0) = f_1(x)$$
$$u(x,b) = f_2(x)$$
$$u(0,y) = 0$$
$$u(a,y) = 0$$



Separation of Variables

Let: $\boldsymbol{u(x,y) = X(x)\, Y(y)}$

Substitute in 2D Laplace PDE: $X''(x)Y(y) + X(x)Y''(x) = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k = constant$$

k= constant since X''/X does not depend on y , and Y"$/Y$ does not depend on x, thus $\frac{\partial}{\partial x}$ $and \frac{\partial}{\partial y}$ $of\ RHS$ *& LHS is zero*

$$\boldsymbol{X'' + k\,X = 0 \quad , Y'' - k\,Y = 0}$$

For the Differential Eqn.: $F''(x) = mF(x)$

$$F(x) = C_1\, e^{\sqrt{m}x} + C_2 e^{-\sqrt{m}x}$$

BC: $X(0) = 0,\ X(a) = 0$ -> $k = \lambda^2 \rightarrow X'' + \lambda^2 X = 0$

$$X_n(x) = \sin\left(\frac{n\pi}{a}x\right) = \sin(\lambda_n x)\,, \quad n = \pm 1,2,3\ldots$$

$$X(x) = \sum_{n=1}^{\infty} X_n(x) = \sum_{n=1}^{\infty} \sin(\lambda_n x)\,, \qquad \lambda_n = \frac{n\pi}{a}$$

$Y'' - \lambda^2 Y = 0$

$$Y(y) = D_1 e^{\lambda y} + D_2 e^{-\lambda y}$$

Using BC: $F(0) = 0 = C_1 + C_2$

Using: $\cosh\theta = 1/2(e^{\theta} + e^{-\theta})$, $\sinh\theta = 1/2(e^{\theta} - e^{-\theta})$

$$\rightarrow \; Y_n(y) = A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)$$

Using Superposition of independent solutions of ODEs:

$$Y(y) = \sum_{n=1}^{\infty} Y_n(y) = \sum_{n=1}^{\infty} A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)$$

$$u(x, y) = X(x)Y(y) = \sum_{n=1}^{\infty} Y_n(y)\, X_n(x)$$

$$= \sum_{n=1}^{\infty} [A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)] \sin(\lambda_n x)$$

Fourier Series in PDE

$$u(x,0) = f_1(x) = \sum_{n=1}^{\infty} [A_n \cosh(\lambda_n 0) + B_n \sinh(\lambda_n 0)] \sin(\frac{n\pi}{a} x)$$

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) = f_1(x)$$

Using Fourier Series: $A_n = \frac{2}{a} \int_0^a f_1(x) \sin\left(\frac{n\pi}{a} x\right) dx$

Fourier Series in PDE (2):

$$u(x,b) = f_2(x)$$

$$\sum_{n=1}^{\infty} [A_n \cosh(\lambda_n b) + B_n \sinh(\lambda_n b)] \sin\left(\frac{n\pi}{a} x\right) = f_2(x)$$

Let $C_n = A_n \cosh(\lambda_n b) + B_n \sinh(\lambda_n b)$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} x\right) = f_2(x)$$

Using Fourier Series: $C_n = \frac{2}{a} \int_0^a f_2(x) \sin\left(\frac{n\pi}{a} x\right) dx$

$$B_n = \frac{C_n - A_n \cosh(\lambda_n b)}{\sinh(\lambda_n b)}$$

Final Solution for a 2D Laplace Equation in a Rectangle Under BC:

$$u(x,0) = f_1(x),\ u(x,b) = f_{2(x)},\ u(0,y) = 0,\ u(a,y) = 0$$

$$u(x,y) = \sum_{n=1}^{\infty} [A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)] \sin(\lambda_n x)$$

$$= \sum_{n=1}^{\infty} \left[A_n \cosh(\lambda_n y) + \frac{C_n - A_n \cosh(\lambda_n b)}{\sinh(\lambda_n b)} \sinh(\lambda_n y)\right] \sin(\lambda_n x)$$

$$= \sum_{n=1}^{\infty} \left[C_n \frac{\sinh(\lambda_n y)}{\sinh(\lambda_n b)} + A_n \left[\cosh(\lambda_n y) - \frac{\cosh(\lambda_n b)}{\sinh(\lambda_n b)} \sinh(\lambda_n y)\right]\right] \sin(\lambda_n x)$$

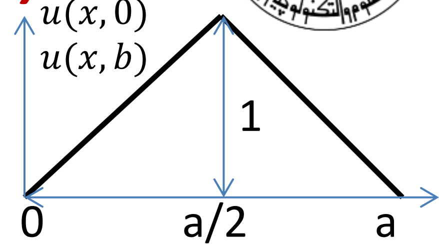
$$u(x,y) = \sum_{n=1}^{\infty} \left[C_n \frac{\sinh(\lambda_n y)}{\sinh(\lambda_n b)} + A_n \frac{\sinh(\lambda_n (b-y))}{\sinh(\lambda_n b)}\right] \sin(\lambda_n x)$$

(using: $\sinh(A-B) = \sinh(A)\cosh(B) - \cosh(A)\sinh(B)$)

Example for 2D Potential in Rectangle:

Triangular Boundary Condition for *u(x,0) & u(x,b)*

$$u(x,0) = u(x,b) = \begin{cases} \frac{2x}{a} & for \;\; 0 < x < \frac{a}{2} \\ \left(2 - \frac{2x}{L}\right) & for \;\; \frac{a}{2} < x < a \end{cases}$$



$$f_1(x) = f_2(x)$$

$$A_n = \frac{2}{a}\int_0^a f_1(x)\sin\left(\frac{n\pi}{a}x\right)dx \;\; C_n = \frac{2}{a}\int_0^a f_2(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

$$A_n = C_n = \frac{2}{a}\int_0^{a/2}\frac{2x}{a}\sin\left(\frac{n\pi}{a}x\right)dx + \frac{2}{a}\int_{a/2}^{a}\left(2 - \frac{2x}{a}\right)\sin\left(\frac{n\pi}{a}x\right)dx$$

$$A_n = C_n = \frac{8}{\pi^2}\left[\sin\left(\frac{n\pi}{2}\right)/n^2\right] \rightarrow Find\, B_n$$