



Power Series Solution of Differential Equations

Lecture 7

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Outline

- Power series Representation of Functions
 - Conditions for Series Convergence
 - Example functions: *exp*, *sin*, *cos*
- Solution of Differential Equations (DEs) using Power Series:
 - 1st Order DE
 - 2nd Order DE

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References

- G. F. Simmons and S. G. Krantz, “Differential Equations: Theory, Technique, and Practice”, McGraw-Hill, 2007
- R. Bronson, “ Differential in Equations”, 2nd Ed., Schaum’s Outlines Series, McGraw-Hill, 1994
- E. Kreyszig, “Advanced Engineering Mathematics”, 8th Ed., 1999

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Power Series Representation of Functions

Using Taylor's series:

$$f(x) = \sum_{j=0}^n \frac{f^{(j)}(0)}{j!} x^j + R_n(x)$$

Remainder:

$$R_n(x) = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} x^{n+1}, \quad \varepsilon \text{ is a number between 0 and } x$$

$$\text{Where: } f^{(j)} = \frac{d^j f(x)}{dx^j}$$

- The series converges if $R_n(x)$ goes to zero
→ n terms are good enough



Test of Convergence for Power Series

If the following limit exists:

$$\lim_{j \rightarrow \infty} \sum_{j=0}^k a_j x^j$$

Another check:

$$\lim_{j \rightarrow \infty} \frac{a_{j+1} x^{j+1}}{a_j x^j}$$

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Example of Common Functions

$$e^x = \sum_{j=0}^{\infty} \frac{f'(0)}{j!} x^j = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin(x) = \sum_{j=0}^{\infty} \frac{f'(0)}{j!} x^j = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cos(x) = \sum_{j=0}^{\infty} \frac{f'(0)}{j!} x^j = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$



Solution of Differential Equations using Power Series

1. Assume: $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$
 $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$
 $y'' = 2a_2 + (3)(2)a_3x + (4)(3)a_4x^2 + \dots$

.....

2. Substitute in differential equation
3. Equate coefficients of x in both sides of differential equation
3. Find coefficients in sequence, starting with a_0 , then a_1 , ...

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Solution of 1st Order Differential Equations using Power Series

Example: $y' = y$

1. Assume: $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$
 $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$

2. Substitute in differential equation

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

3. Equate coefficients of x in both sides of differential equation

$$a_0 = a_1, \quad a_1 = 2a_2, \quad a_2 = 3a_3, \quad \dots$$

4. Find coefficients in sequence, starting with a_0 , then a_1, \dots

If $a_0 = C$ (For example from boundary conditions):

$$a_1 = C, \quad a_2 = \frac{C}{2}, \quad a_3 = \frac{a_2}{3} = \frac{C}{(3)(2)}, \dots, \rightarrow a_n = \frac{C}{n!}$$

$$y = \sum_{j=0}^{\infty} \frac{C}{j!} x^j = Ce^x$$



Solution of 2nd Order Differential Equations using Power Series (1)

Example: $y'' + y = 0$

1. Assume: $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots = \sum_{j=0}^{\infty} a_j x^j$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y'' = 2a_2 + (3)(2)a_3x + (4)(3)a_4x^2 + \dots = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$$

2. Substitute in differential equation

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + \sum_{j=0}^{\infty} a_j x^j = 0$$

Equating start of summation indices:

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + \sum_{j=2}^{\infty} a_{j-2} x^{j-2} = \sum_{j=2}^{\infty} [j(j-1) a_j + a_{j-2}] x^{j-2} = 0$$

3. Equate coefficients of x in both sides of differential equation

$$j(j-1) a_j + a_{j-2} = 0, \text{ for } j = 2, 3, \dots$$



Solution of 2nd Order Differential Equations using Power Series (2)

Example: $y'' + y = 0$

3. Equate coefficients of x in both sides of differential equation

$$j(j-1)a_j + a_{j-2} = 0, \text{ for } j = 2, 3, \dots$$

4. Find coefficients in sequence, starting with a_0 , then a_1, \dots

Assume (from Boundary conditions): $a_0 = A, a_1 = B$

$$j=2: a_2 = \frac{-A}{(2)(1)}, \quad j=4: a_4 = \frac{A}{(4)(3)(2)(1)}$$

$$\rightarrow a_{2j} = (-1)^j \frac{A}{(2j!)} \quad j=1, 2, \dots$$

$$j=3: a_3 = \frac{-B}{(3)(2)(1)}, \quad j=5: a_5 = \frac{B}{(5)(4)(3)(2)(1)}$$

$$\rightarrow a_{2j+1} = (-1)^j \frac{B}{(2j+1)!} \quad j=1, 2, \dots$$



Solution of 2nd Order Differential Equations using Power Series (3)

Example:

$$y'' + y = 0$$

$$y = A \sum_{j=0}^{\infty} (-1)^j \frac{1}{(2j)!} x^{2j} + B \sum_{j=0}^{\infty} (-1)^j \frac{1}{(2j+1)!} x^{2j+1}$$

$$y = A \cos(x) + B \sin(x)$$

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