Finite Difference Method

Lecture 9

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Outline

- Finite difference Method
- Liebmann Method
- Example

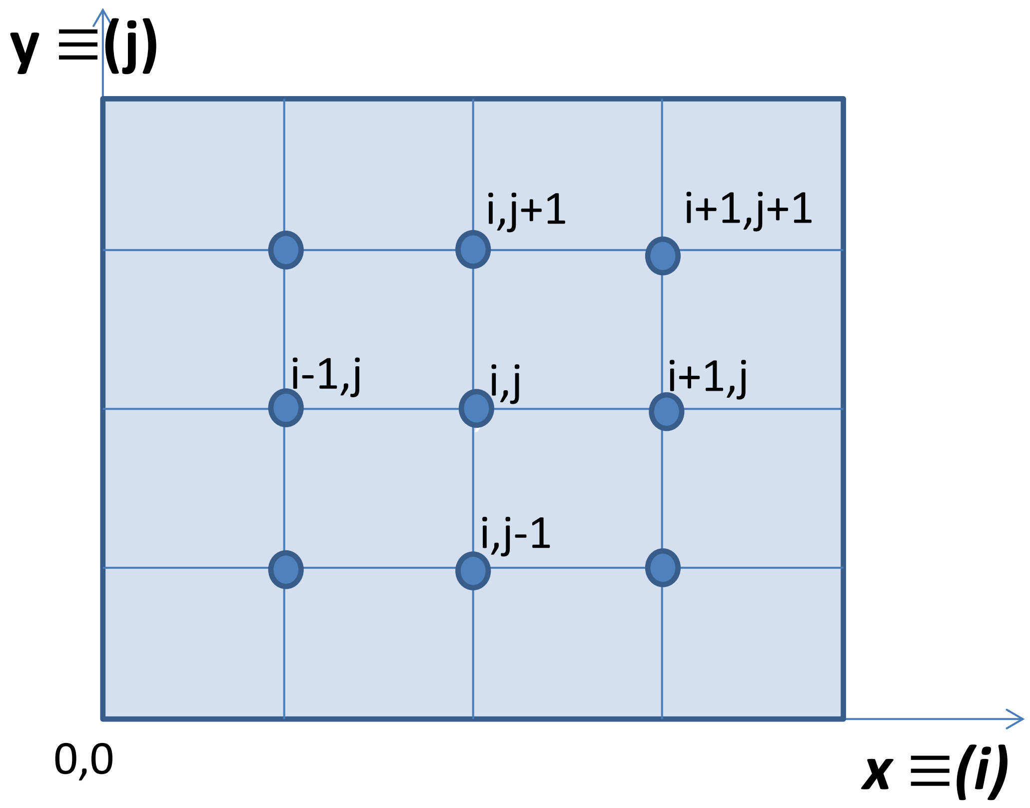
Reference

- S. Chapra and R. Canale, “Numerical Method’s for Engineers”, McGraw-Hill, 5th Ed., 2006

Finite Difference Method

- Solve difference equations on nodes of a grid

2D Grid



2D Laplacian Difference Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}, \qquad \frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,\,j-1}}{\Delta y^2}$$

Assuming $\Delta x = \Delta y = h$

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{h^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,\,j-1}}{h^2} = 0$$

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,\,j-1} - 4T_{i,j} = 0$$

Liebmann Method
(*Guass-Seidel* applied to DE)

Iterate use the latest values in the equation:

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,\,j-1}}{4}$$

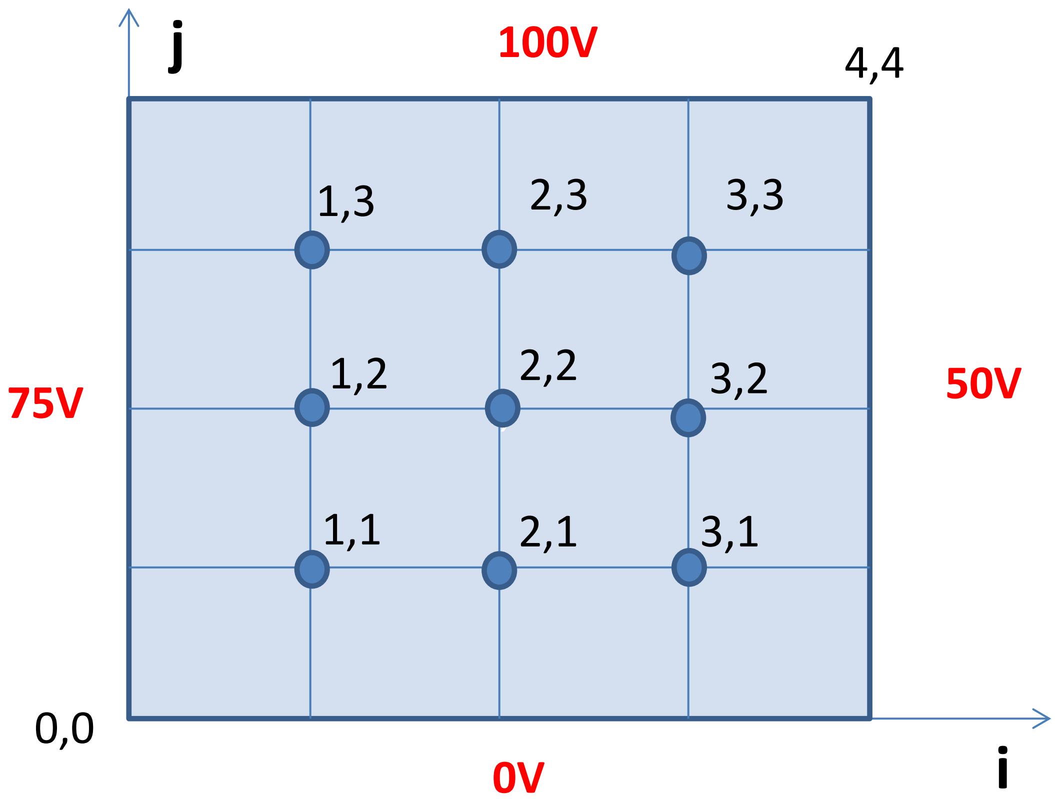
Over-relaxation could be used to accelerate convergence:

$$Y_{i,j}^{(next)} = \lambda\, Y_{i,j}^{(current)} + (1-\lambda)\, Y_{i,j}^{(old)}$$

λ= Over-relaxation Coefficient (range 1-2)

Example: 2D Potential Equation with Drichlet Boundary Conditions

Nodes at (0,j) and (i,0) are boundary conditions and do not add equations to matrix.



2D Potential: Equations: 9 unknowns in 9 equations

Example : Node (1,1): $V_{21} + V_{01} + V_{12} + V_{10} - 4\,V_{11} = 0$

$$V_{01} = 75V, \qquad V_{10} = 0V \;\rightarrow\; 4\,V_{11} - V_{12} - V_{21} = 75$$

$$\begin{bmatrix} 4 & -1 & & -1 & & & & & \\ -1 & 4 & -1 & & -1 & & & & \\ & -1 & 4 & -1 & & -1 & & & \\ -1 & & -1 & 4 & -1 & & -1 & & \\ & -1 & & -1 & 4 & -1 & & -1 & \\ & & -1 & & -1 & 4 & & & -1 \\ & & & -1 & & & 4 & -1 & \\ & & & & -1 & & -1 & 4 & -1 \\ & & & & & -1 & & -1 & 4 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \\ V_{31} \\ V_{12} \\ V_{22} \\ V_{32} \\ V_{13} \\ V_{23} \\ V_{33} \end{bmatrix} = \begin{bmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{bmatrix}$$

Assume λ=1.5

Start with some initial solution (e.g. all 0's):

$$V_{1,1}^{(1)} = \frac{V_{2,1} + V_{0,1} + V_{1,2} + V_{1,0}}{4} = \frac{0 + 75 + 0 + 0}{4} = 18.75V$$

$$V_{1,1}^{(2)} = \lambda\, V_{1,1}^{(1)} + (1-\lambda)\, V_{1,1}^{(0)} = (1.5)\,(18.75) + (1-1.5)\,(0)$$
$$= 28.125V$$

Use this value of V_{11} the next equation for $V_{2,1}$:

$$V_{2,1}^{(1)} = \frac{V_{3,1} + V_{1,1} + V_{1,0} + V_{1,2}}{4} = \frac{0 + 28.175 + 0 + 0}{4} = 7.03V$$

$$V_{2,1}^{(2)} = (1.5)\,(7.03) + (1-1.5)\,(0) = 10.54V$$

Use the latest value of $V_{1,1}$ and $V_{2,1}$ in the next equation, till the 9 equations are finished for the first iteration

Start over a new iteration, where for node (1,1): $V_{1,1}^{(1)}$= 28.125V, $V_{1,1}^{(0)}$= 18.75V, $V_{2,1}$=10.54V, …

Exercise: 2x2 Grid

- Repeat the previous example with 2x2 grid instead of 3x3, using Liebmann's method with overrelaxation (λ=1.5)
- Compare using regular Liebmann method without overrelaxation (i.e. λ=1) with the overrelaxation version
- Compare with using LU to directly solve for the 4 values of V's