



Finite Element Method (3): 2D FEM

Lecture 12-13

Dr. Amr Bayoumi

Fall 2014

Advanced Engineering Mathematics (EC760)

Arab Academy for Science and Technology - Cairo



Outline

- 2D using Triangular Elements
- 2D using Rectangular Elements



References

- S. Chapra and R. Canale, “Numerical Method’s for Engineers”, McGraw-Hill, 5th Ed., 2006
- S. Moaveni, “Finite Element Analysis, Theory and Application with Ansys”, Pearson Prentice Hall, 3rd Ed., 2008
- E. Thompson, “Introduction to the Finite Element Method: Theory, Programming, and Applications”, Wiley, 2005



Triangular Mesh

$$u(x, y) = a_0 + a_{1,1}x + a_{1,2}y$$

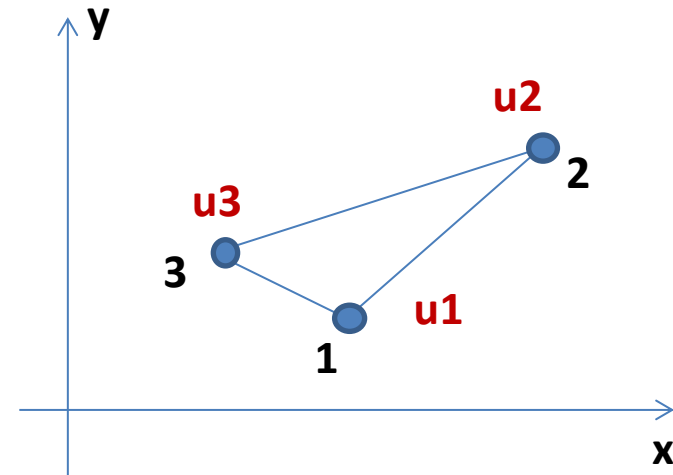
$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_{1,1} \\ a_{1,2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Find a_0 , $a_{1,1}$, $a_{1,2}$ (Cramer's Rule, LU, GE,...)

$$u = N_1u_1 + N_2u_2 + N_3u_3$$

A_e = Area of triangular element = $(1/2) \text{Det}(A)$

- $N_1 = \frac{1}{2A_e} [(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$
- $N_2 = \frac{1}{2A_e} [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$
- $N_3 = \frac{1}{2A_e} [(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$

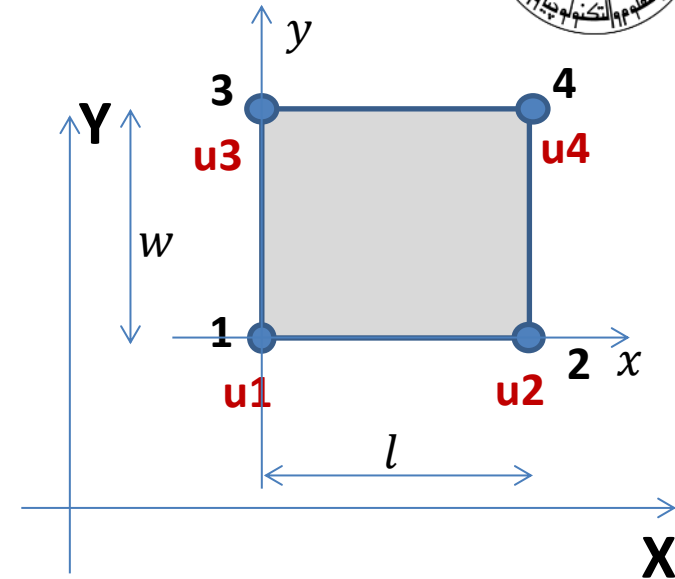


Rectangular Mesh (Local Coordinates)



$$u(x, y) = a_0 + a_1x + a_2y + a_3xy$$

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$



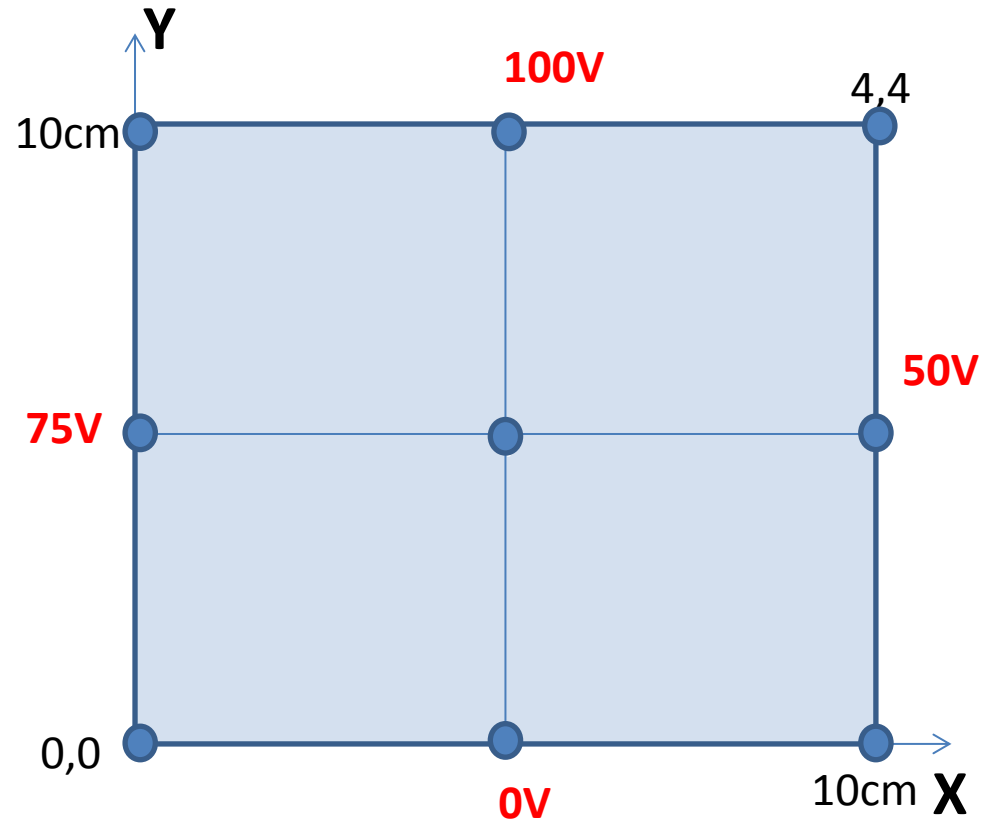
$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

$$N_1 = \left(1 - \frac{x}{l}\right) \left(1 - \frac{y}{w}\right),$$

$$N_2 = \frac{x}{l} \left(1 - \frac{y}{w}\right),$$

$$N_3 = \frac{y}{w} \left(1 - \frac{x}{l}\right), \quad N_4 = \frac{xy}{lw}$$

Example: 2D Potential Equation with Dirichlet Boundary Conditions



Example: 2D Potential Equation with Dirichlet Boundary Conditions



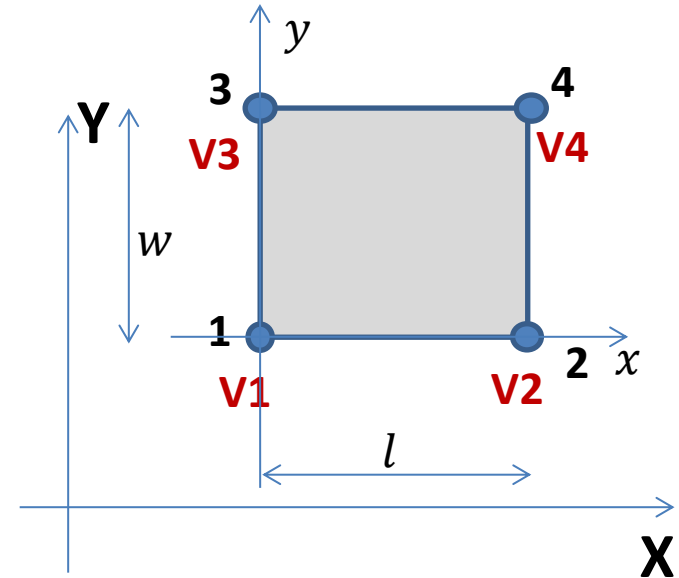
Element Equations:

$$\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = -f(x)$$

$$f(x) = 0$$

$$\frac{\partial V}{\partial x} = -E_x$$

$$\frac{\partial V}{\partial y} = -E_y$$





Derivatives in 2D: $\frac{\partial V}{\partial x}$

$$V = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4 = [N]^T [V]$$
$$\frac{\partial V}{\partial x} = \frac{\partial [N]^T}{\partial x} [V] = \left\{ \frac{\partial}{\partial x} [N_1 \quad N_2 \quad N_3 \quad N_4] \right\} [V]$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x} \left[\left(1 - \frac{x}{l}\right) \left(1 - \frac{y}{w}\right) \right] = \left(1 - \frac{y}{w}\right) \left(\frac{-1}{l} \right) = \frac{y - w}{wl}$$

$$\frac{\partial N_2}{\partial x} = \frac{\partial}{\partial x} \left[\frac{x}{l} \left(1 - \frac{y}{w}\right) \right] = \left(1 - \frac{y}{w}\right) \left(\frac{1}{l} \right) = \frac{w - y}{wl}$$

$$\frac{\partial N_3}{\partial x} = \frac{\partial}{\partial x} \left[\frac{y}{w} \left(1 - \frac{x}{l}\right) \right] = \frac{-y}{wl}, \quad \frac{\partial N_4}{\partial x} = \frac{\partial}{\partial x} \left[\frac{xy}{lw} \right] = \frac{y}{wl}$$

$$\frac{\partial V}{\partial x} = \frac{1}{wl} [(y - w) \quad (w - y) \quad (-y) \quad (y)] [V]$$



Derivatives in 2D: $\frac{\partial V}{\partial y}$

$$\frac{\partial N_1}{\partial y} = \frac{\partial}{\partial y} \left[\left(1 - \frac{x}{l}\right) \left(1 - \frac{y}{w}\right) \right] = \left(1 - \frac{x}{l}\right) \left(\frac{-1}{w}\right) = \frac{x - l}{wl}$$

$$\frac{\partial N_2}{\partial y} = \frac{\partial}{\partial y} \left[\frac{x}{l} \left(1 - \frac{y}{w}\right) \right] = \frac{-x}{wl}$$

$$\frac{\partial N_3}{\partial y} = \frac{\partial}{\partial y} \left[\frac{y}{w} \left(1 - \frac{x}{l}\right) \right] = \frac{l - x}{wl}, \quad \frac{\partial N_4}{\partial x} = \frac{\partial}{\partial x} \left[\frac{xy}{lw} \right] = \frac{x}{wl}$$

$$\frac{\partial V}{\partial y} = \frac{1}{wl} [(x - l) \quad (l - x) \quad (-x) \quad (x)][V]$$



Residual Equations in 2D Using Galerkin's Method

$$\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = 0$$
$$R_i = \int_A N_i \left(\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} \right) dx dy = 0$$

Use:

$$\frac{\partial}{\partial x} \left([N]^T \frac{\partial V}{\partial x} \right) = [N]^T \frac{\partial^2 V}{\partial x^2} + \frac{\partial [N]^T}{\partial x} \frac{\partial V}{\partial x}$$