Eigenvalue Problems: Iterative Methods

Lecture 3

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Outline

- Iterative Solutions:
  - Highest Eigenvalue: Power Method
  - Lowest Eigenvalue: Inverse Power Method
  - Other Eigenvalues: Eigenvalue Substitution

Power Method: Dominant Eigenvalue

Target: $\mathbf{y} = \mathrm{A}\mathbf{x} = \lambda\mathbf{x}$

Start with all 1's **x** vector: $\mathbf{x_0}=[1\ 1\ \ldots\ 1]^T$

$\boldsymbol{y}^{(1)} = \mathrm{A}\boldsymbol{x}^{(0)} = \lambda^{(1)}\boldsymbol{x}^{(1)}$ (*Iteration number = 1*)

$\lambda^{(1)}$= element in $\mathbf{y}_1$ with highest absolute value

$$\boldsymbol{x}^{(1)} = \frac{1}{\lambda^{(1)}}\ \boldsymbol{y}^{(1)} = \frac{1}{\lambda^{(1)}}\mathrm{A}\boldsymbol{x}^{(0)}$$

$\boldsymbol{y}^{(2)} = \mathrm{A}\boldsymbol{x}^{(1)} = \frac{1}{\lambda^{(1)}}\mathrm{A}^2\boldsymbol{x}^{(0)}$ (*Iteration number = 2*)

$\lambda^{(2)}$= element in $\mathbf{y}_2$ with highest absolute value

$\boldsymbol{x}^{(2)} = \frac{1}{\lambda^{(2)}}\ \boldsymbol{y}^{(2)} = \frac{1}{\lambda^{(2)}\lambda^{(1)}}\mathrm{A}^2\boldsymbol{x}^{(0)}$……

$$\boldsymbol{y}^{(k)} = \mathrm{A}\boldsymbol{x}^{(k-1)} \rightarrow \boldsymbol{x}^{(k)} = \frac{1}{\lambda^{(k)}}\ \boldsymbol{y}^{(k)} \rightarrow \boldsymbol{x}^{(k)} \approx \frac{1}{\lambda^{k}}\mathrm{A}^k\boldsymbol{x}^{(0)}$$

Power Method: Dominant Eigenvalue Proof

Assume $\mathbf{x} = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \ldots.. + d_n\mathbf{v_n}$

Where $\mathbf{v_n}$ are linearly independent eigenvectors

$$\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\mathrm{A}\mathbf{x} = \lambda_1 d_1\mathbf{v}_1 + \lambda_2 d_2\mathbf{v}_2 + \ldots.. + \lambda_n d_n\mathbf{v}_n$$

$$\mathrm{A}^2\mathbf{x} = \lambda_1{}^2 d_1\mathbf{v}_1 + \lambda_2{}^2 d_2\mathbf{v}_2 + \ldots.. + \lambda_n{}^2 d_n\mathbf{v}_n$$

After k iterations:

$$\mathrm{A}^k\mathbf{x} = \lambda_1{}^k d_1\mathbf{v}_1 + \lambda_2{}^k d_2\mathbf{v}_2 + \ldots.. + \lambda_n{}^k d_n\mathbf{v}_n$$

$$\frac{1}{\lambda_1{}^k}\mathrm{A}^k\mathbf{x} = d_1\mathbf{v}_1 + (\lambda_2/\lambda_1)^k d_2\mathbf{v}_2 \ldots + (\lambda_n/\lambda_1)^k d_n\mathbf{v}_n$$

If λ_1 is considerably higher than λ_2 , ..., λ_n: $\frac{1}{\lambda_1{}^k}\mathrm{A}^k\mathbf{x} \to d_1\mathbf{v}_1$

Example on Highest Eigenvalue

(*in Magnitude, sign ignored*)

$$y = Ax = \begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda x$$

Exact Det. method: $\lambda_1 = 9.38, \lambda_2 = -1.38$

Iter. 1: $y^{(1)} = Ax^{(0)} = \begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 0.9 \end{bmatrix} \rightarrow \lambda^{(1)} = 10$

Iter. 2: $y^{(2)} = Ax^{(1)} = \begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 9.3 \\ 8.5 \end{bmatrix} = 9.3 \begin{bmatrix} 1 \\ 0.914 \end{bmatrix}$

$\rightarrow \lambda^{(2)} = 9.3$

$y^{(3)} = Ax^{(2)} = \begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.914 \end{bmatrix} = \begin{bmatrix} 9.39 \\ 8.56 \end{bmatrix} = 9.39 \begin{bmatrix} 1 \\ 0.9116 \end{bmatrix}$

$\rightarrow \lambda^{(3)} = 9.39$

$y^{(4)} = Ax^{(3)} = \begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9116 \end{bmatrix} = \begin{bmatrix} 9.381 \\ 8.558 \end{bmatrix} = 9.381 \begin{bmatrix} 1 \\ 0.912 \end{bmatrix}$

$\rightarrow \lambda^{(4)} = 9.381, \; ....$

Inverse Power Method: Smallest Absolute Eigenvalue

Target: $\mathbf{y} = A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x} = \propto \mathbf{x}$

$$B\mathbf{x} = \propto \mathbf{x}$$

At iteration k: $\frac{B^k}{\alpha_1{}^k}\mathbf{x}$ **--->** $d_1\, \mathbf{v}_1$

Dominant $\propto$ is equivalent to smallest absolute λ

Use LU factorization to solve for y:

$$A\mathbf{y}^{(k)} = \mathbf{x}^{(k-1)}$$

Find dominant element in y(k) as $\propto$

Keep on, then least $\lambda = \propto^{-1}$

Example on Inverse Power Method: lowest Eigenvalue

$$y = A^{-1}x = \lambda^{-1}x = \alpha x,$$

$$A = \begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix} = [L][U] = \begin{bmatrix} 3 & 0 \\ 4 & -4.333 \end{bmatrix} \begin{bmatrix} 1 & 2.333 \\ 0 & 1 \end{bmatrix}$$

Exact Det. method: $\lambda_1 = 9.38, \lambda_2 = -1.38 \rightarrow \alpha_1 = 1/\lambda_2 = -0.7246$

Using LU factorization (only once), forward/backwards substitution:

$$Ay^{(1)} = x^{(0)} \rightarrow Ay^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} y_1{}^{(1)} \\ y_2{}^{(1)} \end{bmatrix} = \begin{bmatrix} 0.1538 \\ 0.0769 \end{bmatrix} = 0.1538 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$\rightarrow \alpha^{(1)} = 0.1538$

$$Ay^{(2)} = x^{(1)} \rightarrow Ay^{(2)} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \rightarrow \begin{bmatrix} y_1{}^{(2)} \\ y_2{}^{(2)} \end{bmatrix} = \begin{bmatrix} -0.11538 \\ 0.192307 \end{bmatrix} = 0.192307 \begin{bmatrix} -0.6 \\ 1 \end{bmatrix}$$

$\rightarrow \alpha^{(2)} = -0.6538$

$$Ay^{(3)} = x^{(3)} \rightarrow Ay^{(3)} = \begin{bmatrix} -0.6 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} y_1{}^{(3)} \\ y_2{}^{(3)} \end{bmatrix} = \begin{bmatrix} 0.7692 \\ -0.4153 \end{bmatrix} = 0.7692 \begin{bmatrix} 1 \\ -0.54 \end{bmatrix}$$

$\rightarrow \alpha^{(3)} = -0.733$

$$Ay^{(4)} = x^{(3)} \rightarrow Ay^{(4)} = \begin{bmatrix} 1 \\ -0.54 \end{bmatrix} \rightarrow \begin{bmatrix} y_1{}^{(4)} \\ y_2{}^{(4)} \end{bmatrix} = \begin{bmatrix} -0.7292 \\ 0.4554 \end{bmatrix} = -0.7292 \begin{bmatrix} 1 \\ 0.6245 \end{bmatrix}$$

$$\rightarrow \boldsymbol{\alpha_1{}^{(4)} = -0.729 -> \lambda_2 = -1.371, \ldots}$$

Power Method: Rest of Eigenvalues

- Find one eigenvalue using power method, λ_1
- Use it to shift a new set of diagonals, such that

  $\lambda' = \lambda - \lambda_1$

- Solve for the next eigenvalue in:

$\mathbf{y} = C\mathbf{x} = [\mathbf{A} - \lambda_1 \mathbf{I}]\mathbf{x}$ **(Power Method: Most Dominant)**

One eigenvalue is 0 (previously found, shifted)

$\lambda_2 = \lambda'_1 + \lambda_1$