



Boundary Value Problems in Partial Differential Equations:

2-D Laplace Equation

(Potential Equation)

Lecture 5

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Outline

- 2D Laplace Equation
- Boundary Conditions:
 - Dirichlet BC
 - Neumann BC
 - Mixed BC
- Separation of Variables
- Fourier Series Analysis

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References

- D.L. Powers, “Boundary Value Problems and Partial Differential Equations”, Academic Press, 6th Ed., 2010

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2D Laplace Equation (PDE)

$$\nabla^2 u(x, y) = \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

Example: Electrostatic potential in 2D

Poisson's Equation:

$$\nabla^2 u(x, y) = -H(x, y)$$

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Boundary Conditions

- Dirichlet's BC:

$u(x, y)$ predefined along boundaries

- Neumann's BC:

$\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ predefined along boundaries

- Mixed BCs:

Mix of $u(x, y)$ and $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial y}$ predefined along boundaries

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Example Boundary Conditions: 2D Potential in a Rectangle

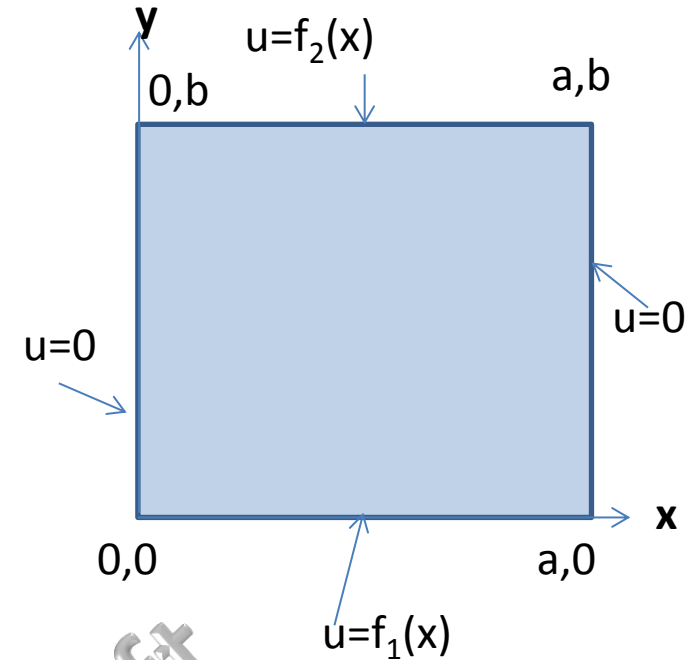
- Dirichlet's BC:

$$u(x, 0) = f_1(x)$$

$$u(x, b) = f_2(x)$$

$$u(0, y) = 0$$

$$u(a, y) = 0$$



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Separation of Variables

Let: $u(x, y) = X(x) Y(y)$

Substitute in 2D Laplace PDE: $X''(x)Y(y) + X(x)Y''(y) = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k = \text{constant}$$

$k = \text{constant}$ since X''/X does not depend on y , and Y''/Y does not depend on x , thus $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of RHS & LHS is zero

$$X'' + kX = 0, \quad Y'' - kY = 0$$

For the Differential Eqn.: $F''(x) = mF(x)$

$$F(x) = C_1 e^{\sqrt{m}x} + C_2 e^{-\sqrt{m}x}$$

BC: $X(0) = 0, X(a) = 0 \rightarrow k = \lambda^2 \rightarrow X'' + \lambda^2 X = 0$

$$X_n(x) = \sin\left(\frac{n\pi}{a}x\right) = \sin(\lambda_n x), \quad n = \pm 1, 2, 3, \dots$$

$$X(x) = \sum_{n=1}^{\infty} X_n(x) = \sum_{n=1}^{\infty} \sin(\lambda_n x), \quad \lambda_n = \frac{n\pi}{a}$$



$$Y'' - \lambda^2 Y = 0$$

$$Y(y) = D_1 e^{\lambda y} + D_2 e^{-\lambda y}$$

Using BC: $F(0) = 0 = C_1 + C_2$

Using: $\cosh \theta = 1/2(e^\theta + e^{-\theta})$, $\sinh \theta = 1/2(e^\theta - e^{-\theta})$
 $\rightarrow Y_n(y) = A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)$

Using Superposition of independent solutions of ODEs:

$$Y(y) = \sum_{n=1}^{\infty} Y_n(y) = \sum_{n=1}^{\infty} A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)$$

$$u(x, y) = X(x)Y(y) = \sum_{n=1}^{\infty} Y_n(y) X_n(x)$$
$$= \sum_{n=1}^{\infty} [A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)] \sin(\lambda_n x)$$



Fourier Series in PDE

$$u(x, 0) = f_1(x) = \sum_{n=1}^{\infty} [A_n \cosh(\lambda_n 0) + B_n \sinh(\lambda_n 0)] \sin\left(\frac{n\pi}{a} x\right)$$
$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) = f_1(x)$$

Using Fourier Series: $A_n = \frac{2}{a} \int_0^a f_1(x) \sin\left(\frac{n\pi}{a} x\right) dx$

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Fourier Series in PDE (2):

$$u(x, b) = f_2(x)$$

$$\sum_{n=1}^{\infty} [A_n \cosh(\lambda_n b) + B_n \sinh(\lambda_n b)] \sin\left(\frac{n\pi}{a} x\right) = f_2(x)$$

Let $C_n = A_n \cosh(\lambda_n b) + B_n \sinh(\lambda_n b)$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} x\right) = f_2(x)$$

Using Fourier Series: $C_n = \frac{2}{a} \int_0^a f_2(x) \sin\left(\frac{n\pi}{a} x\right) dx$

$$B_n = \frac{C_n - A_n \cosh(\lambda_n b)}{\sinh(\lambda_n b)}$$



Final Solution for a 2D Laplace Equation in a Rectangle Under BC:

$$u(x, 0) = f_1(x), u(x, b) = f_2(x), u(0, y) = 0, u(a, y) = 0$$

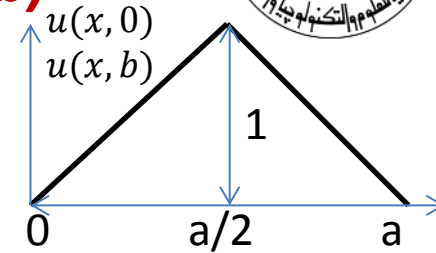
$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} [A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)] \sin(\lambda_n x) \\ &= \sum_{n=1}^{\infty} \left[A_n \cosh(\lambda_n y) + \frac{C_n - A_n \cosh(\lambda_n b)}{\sinh(\lambda_n b)} \sinh(\lambda_n y) \right] \sin(\lambda_n x) \\ &= \sum_{n=1}^{\infty} \left[C_n \frac{\sinh(\lambda_n y)}{\sinh(\lambda_n b)} + A_n \left[\cosh(\lambda_n y) - \frac{\cosh(\lambda_n b)}{\sinh(\lambda_n b)} \sinh(\lambda_n y) \right] \right] \sin(\lambda_n x) \\ u(x, y) &= \sum_{n=1}^{\infty} \left[C_n \frac{\sinh(\lambda_n y)}{\sinh(\lambda_n b)} + A_n \frac{\sinh(\lambda_n (b - y))}{\sinh(\lambda_n b)} \right] \sin(\lambda_n x) \end{aligned}$$

(using: $\sinh(A - B) = \sinh(A)\cosh(B) - \cosh(A)\sinh(B)$)



Example for 2D Potential in Rectangle: Triangular Boundary Condition for $u(x,0)$ & $u(x,b)$

$$u(x,0) = u(x,b) = \begin{cases} \frac{2x}{a} & \text{for } 0 < x < \frac{a}{2} \\ \left(2 - \frac{2x}{L}\right) & \text{for } \frac{a}{2} < x < a \end{cases}$$



$$f_1(x) = f_2(x)$$

$$A_n = \frac{2}{a} \int_0^a f_1(x) \sin\left(\frac{n\pi}{a}x\right) dx \quad C_n = \frac{2}{a} \int_0^a f_2(x) \sin\left(\frac{n\pi}{a}x\right) dx$$

$$A_n = C_n = \frac{2}{a} \int_0^{a/2} \frac{2x}{a} \sin\left(\frac{n\pi}{a}x\right) dx + \frac{2}{a} \int_{a/2}^a \left(2 - \frac{2x}{a}\right) \sin\left(\frac{n\pi}{a}x\right) dx$$

$$A_n = C_n = \frac{8}{\pi^2} \left[\sin\left(\frac{n\pi}{2}\right) / n^2 \right] \rightarrow \text{Find } B_n$$

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