Power Series Solution of Differential Equations

Lecture 6

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Outline

- Power series Representation of Functions
  - Conditions for Series Convergence
  - Example functions: *exp, sin, cos*
- Solution of Differential Equations (DEs) using Power Series:
  - 1^{st} Order DE
  - 2^{nd} Order DE

References

- G. F. Simmons and S. G. Krantz, “Differential Equations: Theory, Technique, and Practice”, McGraw-Hill, 2007
- R. Bronson, “ Differential in Equations”, 2nd Ed., Schaum’s Outlines Series, McGraw-Hill, 1994
- E. Kreyszig, “Advanced Engineering Mathematics”, 8th Ed., 1999

Power Series Representation of Functions

Using Taylor's series:

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j + Rn(x)$$

Remainder:

$$R_n(x) = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} x^{n+1}\ ,$$ ε is a number between 0 and x

Where: $f^{(j)} = \frac{d^j f(x)}{dx^j}$

- The series converges if $R_n(x)$ goes to zero

$\rightarrow$ n terms are good enough

Test of Convergence for Power Series

If the following limit exists:

$$\lim_{j\to\infty} \sum_{j=0}^{k} a_j x^j$$

Another check:

$$\lim_{j\to\infty} \frac{a_{j+1}\, x^{j+1}}{a_j x^j} < 1$$

Example of Common Functions

$$e^x = \sum_{j=0}^{n} \frac{f'(0)}{j!} = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin(x) = \sum_{j=0}^{\infty} \frac{f'(0)}{j!} x^j = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = \sum_{j=0}^{\infty} \frac{f'(0)}{j!} x^j = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Solution of Differential Equations using Power Series

1. Assume: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$
$$y'' = 2a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \cdots$$

....

2. Substitute in differential equation
3. Equate coefficients of x in both sides of differential equation
3. Find coefficients in sequence, starting with a_0, then a_1, …

Solution of 1st Order Differential Equations using Power Series

Example: $$y' = y$$

1. Assume: $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

2. Substitute in differential equation

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

3. Equate coefficients of x in both sides of differential equation

$$a_0 = a_1, \qquad a_1 = 2a_2, \qquad a_2 = 3a_3, \qquad \ldots$$

4. Find coefficients in sequence, starting with a_0, then a_1, …

If $a_o = C$ (For example from boundary conditions):

$$a_1 = C, \qquad a_2 = \frac{C}{2}, \qquad a_3 = \frac{a_2}{3} = \frac{C}{(3)(2)}, \ldots, \rightarrow \quad a_n = \frac{C}{n!}$$

$$y = \sum_{j=0}^{\infty} \frac{C}{j!} x^j = Ce^x$$

Solution of 2nd Order Differential Equations using Power Series (1)

Example: $\boldsymbol{y'' + y = 0}$

1. Assume: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = \sum_{j=0}^{\infty} a_j\, x^j$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{j=1}^{\infty} j\, a_j\, x^{j-1}$

$y'' = 2a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \cdots = \sum_{j=2}^{\infty} j\,(j-1)\, a_j\, x^{j-2}$

2. Substitute in differential equation

$$\sum_{j=2}^{\infty} j\,(j-1)\, a_j\, x^{j-2} + \sum_{j=0}^{\infty} a_j\, x^j = 0$$

Equating start of summation indices:

$$\sum_{j=2}^{\infty} j\,(j-1)\, a_j\, x^{j-2} + \sum_{j=2}^{\infty} a_{j-2}\, x^{j-2} = \sum_{j=2}^{\infty} [\, j\,(j-1)\, a_j + a_{j-2}\,]\, x^{j-2} = 0$$

3. Equate coefficients of x in both sides of differential equation

$$j\,(j-1)\, a_j + a_{j-2} = 0, \text{ for } j = 2, 3, \ldots$$

Solution of 2nd Order Differential Equations using Power Series (2)

Example: $\boldsymbol{y'' + y = 0}$

3. Equate coefficients of x in both sides of differential equation

$$j\,(j-1)\,a_j + a_{j-2} = 0, \text{ for } j = 2, 3, \ldots$$

4. Find coefficients in sequence, starting with a_0, then a_1, …

Assume (from Boundary conditions): $a_0 = A, \; a_1 = B$

j=2: $a_2 = \frac{-A}{(2)(1)}$, j=4: $a_4 = \frac{A}{(4)(3)(2)(1)}$

$$\rightarrow a_{2j} = (-1)^j \frac{A}{(2j!)} \qquad j=1,2, \ldots$$

j=3: $a_3 = \frac{-B}{(3)(2)(1)}$, j=5: $a_5 = \frac{B}{(5)(4)(3)(2)(1)}$

$$\rightarrow a_{2j+1} = (-1)^j \frac{B}{(2j+1)!} \qquad j=1,2, ..$$

Solution of 2nd Order Differential Equations using Power Series (3)

Example: $y'' + y = 0$

$$y = A\sum_{j=0}^{\infty}(-1)^j \frac{1}{(2j!)} x^{2j} + B\sum_{j=0}^{\infty}(-1)^j \frac{1}{(2j+1)!} x^{2j+1}$$

$$y = A\cos(x) + B\sin(x)$$