



# Special Functions

Lecture 8

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# Outline

- Singular Points
- The Method of Frobenius
- Bessel Function



# References

- G. F. Simmons and S. G. Krantz, “Differential Equations: Theory, Technique, and Practice”, McGraw-Hill, 2007
- D.L. Powers, “Boundary Value Problems and Partial Differential Equations”, Academic Press, 6<sup>th</sup> Ed., 2010
- R. Bronson, “ Differential in Equations”, 2<sup>nd</sup> Ed., Schaum’s Outlines Series, McGraw-Hill, 1994
- E. Kreyszig, “Advanced Engineering Mathematics”, John Wiley & Sons



# Frobenius Method

- Find a solution for the 2<sup>nd</sup> order differential equation:

$$y'' + \frac{P(x)}{x} y' + \frac{Q(x)}{x^2} y = 0$$

Multiplying by  $x^2$ :  $x^2 y'' + xP(x)y' + Q(x)y = 0$

**Frobenius Method:** A series solution can be found near  $x_0 = 0$  as:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

If **Taylor's expansion**  $\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$  at  $x_0 = 0$  is **Analytical** for both:

$$xP(x) \text{ and } Q(x)$$

-> Series Solution of D.E. can be used

$x_0$  is called *Regular Singular Point*

If singularity is around  $x_0 \neq 0$ , make a translation of axis to  $(x - x_0)$



# Frobenius Method: Procedure

$$x^2y'' + P(x)xy' + Q(x)y = 0$$

Frobenius Method:  $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots + a_n x^{r+n} + \dots$$

$$y' = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x^{r+1} + \dots \\ + (r+n) a_n x^{r+n-1} + \dots$$

$$y'' = (r-1) r a_0 x^{r-2} + (r+1) r a_1 x^{r-1} + (r+2)(r+1) a_2 x^r \\ + \dots + (r+n+2)(r+n+1) a_n x^{r+n} + \dots$$

- Substitute in DE and Equate coefficients for  $x$ . This results in:
  - **Indicial Equation** by equating coefficients of  $x^0$ : use to find two values of  $r$
  - **Recurrence Formula** by equating coefficients of  $x^n$



# Frobenius Method: **Indicial Equation**

- A quadratic equation in  $r$
- Used to find  $r_1, r_2$
- Results in two solutions  $y_{r_1}(x)$  and  $y_{r_2}(x)$



# Example Application of Frobenius Method: Bessel's Equation

$$y'' + \frac{1}{x}y' + \left(\frac{x^2 - p^2}{x^2}\right)y = 0 \rightarrow x^2y'' + xy' + (x^2 - p^2)y = 0$$

- $P(x)/x$  and  $Q(x)/x^2$  are singular at  $x=0$
- $xP(x)$  and  $Q(x)$  are analytical at  $x=0$

## ***Example Applications of Bessel's Function:***

- Laplace Eq. in cylindrical coordinates
- Schrodinger's equation in radial form



# Application: Bessel's Equation

## Laplace Eq. in Cylindrical Coordinates,

$$y'' + \frac{1}{x}y' + \left(\frac{x^2 - p^2}{x^2}\right)y = 0 \rightarrow x^2y'' + xy' + (x^2 - p^2)y = 0$$

- Using Frobenius Method for series representation of  $y$ , and substituting in DE, and equating coefficients:

$$n=0: r(r-1)a_0 + ra_0 - p^2a_0 = 0 \quad (\text{Indicial Equation})$$

$$n=1: (r+1)ra_1 + (r+1)a_1 - p^2a_1 = 0$$

$$\rightarrow [(r+1)^2 - p^2]a_1 = 0 \rightarrow a_1 = 0 \quad (\text{since } r+1 = p \text{ cannot be imposed})$$

$$n=s: (s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - p^2a_s = 0$$

$$\text{Indicial Equation: } (r+p)(r-p) = 0 \rightarrow r_1 = p \geq 0, r_2 = -p$$

$$\text{Recurrence Formula: } (s+r+p)(s+r-p) + a_{s-2} = 0$$





# Bessel's Equation: Recurrence Formula

$$r = r_1 = p$$

Recurrence Formula:

$$(s + r + p)(s + r - p)a_s + a_{s-2} = 0$$

For  $r = p$ :  $(s + 2p)s a_s + a_{s-2} = 0$

$a_1 = 0 \rightarrow a_3 = 0, a_5 = 0 \rightarrow$  use only even indices:  $s \Rightarrow 2n$

$$(2n + 2p)(2n) a_{2n} + a_{2n-2} = 0$$

Recurrence Formula: 
$$a_{2n} = \frac{-1}{2^{2n}(p + n)} a_{2n-2}$$

General Formula, in terms of  $a_0$  :

$$a_{2n} = \frac{(-1)^n}{2^{2n} n! (p + 1)(p + 2) \dots (p + n)} a_0$$



# Bessel's Equation: **Bessel's Function of First Kind** for $r = r_1 = p = N = \text{Integer}$

General Formula ( $N = p = 0, 1, 2, \dots$  i. e.  $p^2 = 0, 1, 4, 9, \dots$ ):

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(N+1)(N+2)\dots(N+n)} a_0$$

$a_0$  is an arbitrary coeff., typically expressed as:  $a_0 = \frac{1}{2^N N!}$

$$a_{2n} = \frac{(-1)^n}{2^{2n+N} n! (n + N)!}$$

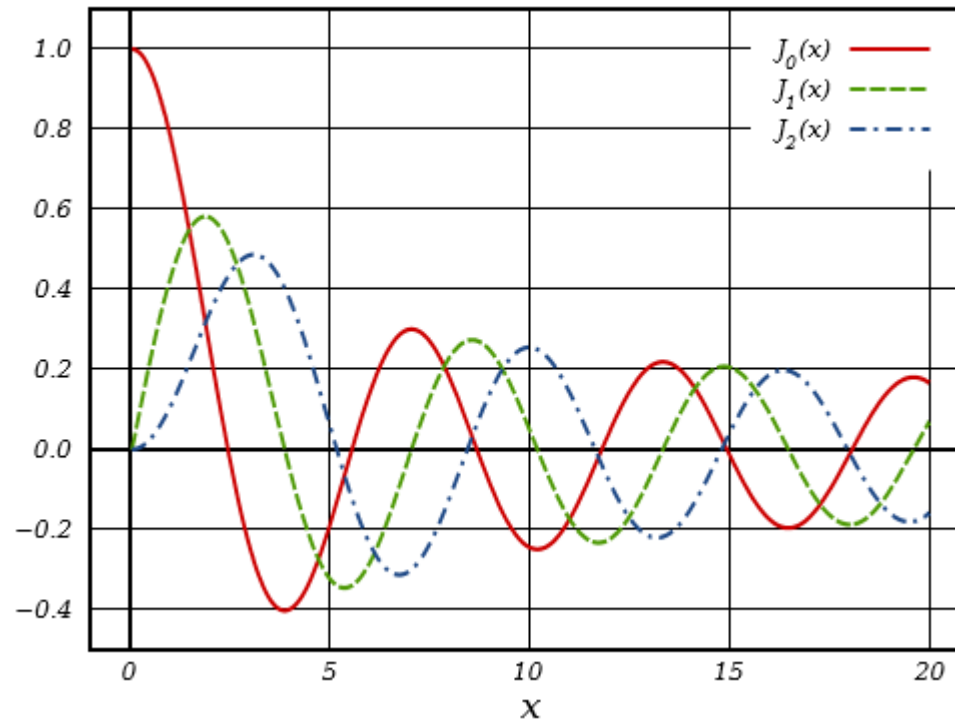
One of the solutions of Bessel's Equation is **Bessel's Function** of **First Kind** of order  $N$ :

$$J_N(x) = x^N \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+N} n! (N + n)!}$$

Where  $N = p = \text{Integer}$



# Plots of $J_N(x)$



Reprinted from Wolfram Mathworld:

<http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html>



# Notes

- $J_N(x)$  converges at  $x=0$
- Solution exists for  $p \neq Integer$  using Gamma Functions
- Another solution exists  $Y(x)$  which does not converge at  $x=0$