Integrals (2):
Gamma Function

Lecture 9

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References

- E. Kreyszig, "Advanced Engineering Mathematics", 8th Ed., 1999
- M.R.Spiegel," Fourier Analysis with Applications to Boundary Value Problems", Schaum's Outlines Series, McGraw-Hill
- R. Bronson, " Differential in Equations", 2nd Ed., Schaum's Outlines Series, McGraw-Hill, 1994

Outline

- Definition of Gamma Functions
- Properties of Gamma Functions
- Integrals using Gamma Functions
- Bessel Functions using Gamma Functions

Definition of Gamma Function $\Gamma(v)$

$$\Gamma(v) = \int_0^\infty e^{-t} t^{v-1}\, dt$$

- $v \neq 0$ can be any positive or negative real or complex number
- Used to get factorials of non-integer numbers
- Evaluated using tables for small v

Special Gamma Function $\Gamma(v)$

- $\Gamma(1) = \int_0^\infty e^{-t}t^0 dt = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 0 - (-1) = 1$
- $\Gamma(0) \rightarrow undefined$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Properties of Gamma Function

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu \, dt = \int_0^\infty t^\nu d(-e^{-t})$$

Integrate by parts: $\Gamma(\nu+1) = \left.(-e^{-t})\, t^\nu\right|_0^\infty - \int_0^\infty (-e^{-t}) d(t^\nu)$

$$= 0 + \nu \int_0^\infty e^{-t} t^{\nu-1} \, dt = \nu\Gamma(\nu)$$

$$\Gamma(\nu+1) = \nu\Gamma(\nu)$$

Properties of Gamma Function (2)

$$\Gamma(\nu+1) = \nu\Gamma(\nu) \to \Gamma(v) = \frac{\Gamma(v+1)}{v}$$

$$\to \Gamma(0) = \frac{\Gamma(0+1)}{0} \to \lim_{v\to 0^+} \frac{1}{0^+} = +\infty, \lim_{v\to 0^-} \frac{1}{0^-} = -\infty$$

For negative $x = v - 1, x \neq 0$:

$$\Gamma(x) = \Gamma(v-1) = \frac{\Gamma(v)}{v-1}$$

$$\to \Gamma(-1) = \Gamma(0-1) = \frac{\Gamma(0)}{0-1} = \frac{\pm\infty}{-1}$$

$$\to \Gamma\left(-\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}-1\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{1}{2}-1} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

Check for example: https://www.physicsforums.com/threads/gamma-function-on-negative-fractions.489613/

Generalization of Factorials using Gamma Function

$$\Gamma(2) = 1\,\Gamma(1) = 1!, \qquad \Gamma(3) = 2\Gamma(2) = 2!$$

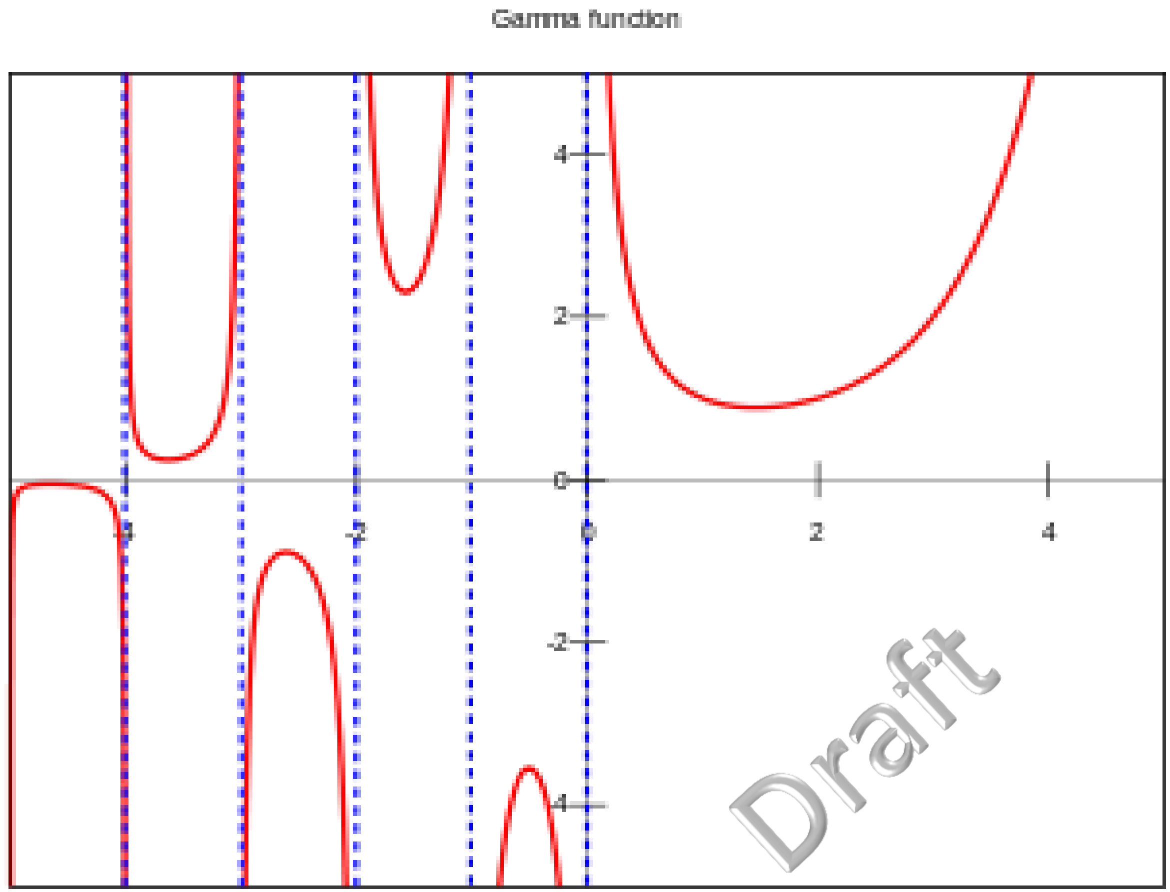
For n=integer: $\Gamma(n+1) = n!$

For any $x = n + v$, where $v > 0$ is not an integer:

$$\Gamma(x) = \Gamma(v+n) = (v+n-1)\Gamma(v+n-1)$$
$$= (v+n-1)(v+n-2)\Gamma(v+n-2)$$
$$\Gamma(v+n) = (v+n-1)(v+n-2)\ldots(v+1)\Gamma(v+1)$$

Example: $\Gamma(4.5) = (3.5)(2.5)(1.5)(0.5)\Gamma(0.5) = 5.5625\sqrt{\pi}$

Plots of $\Gamma(v)$



Source: http://en.wikipedia.org/wiki/Gamma_function

Example Integrals Using $\Gamma(v)$

- Example: Integral used to calculate number of conduction electrons in semiconductors, for Energy $E = E_c$ to $E \to \infty$:

$$I_1 = \int_{E-Ec=0}^{\infty} \sqrt{E - E_c}\, e^{-(E-E_c)} d(E - E_c)$$

Using $x = E - E_c$

$$\to I_1 = \int_0^{\infty} x^{1/2}\, e^{-x} dx = \int_0^{\infty} e^{-x} x^{(\frac{3}{2}-1)} dx = \Gamma\left(\frac{3}{2}\right)$$

$$= \Gamma(1 + 0.5) = 0.5\, \Gamma(0.5) = \sqrt{\pi}/2$$

Reminder: Bessel's Function of First Kind for $r = r_1 = p = N = Integer$

General Formula ($N = p = 0,1,2,\dots\ \ i.e. p^2 = 0,1,4,9,\dots$):

$$a_{2n} = \frac{(-1)^n}{2^{2n}\, n!(N+1)(N+2)\dots(N+n)}\, a_0$$

a_0 is an arbitrary coeff., typically expressed as: $a_0 = \frac{1}{2^N N!}$

$$a_{2n} = \frac{(-1)^n}{2^{2n+N}\, n!\,(n+N)!}$$

One of the solutions of Bessel's Equation is ***Bessel's Function*** of ***First Kind*** of order ***N***:

$$J_N(x) = x^N \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{2^{2n+N}\, n!\,(N+n)!}$$

Where $N = p = Integer$

Application to $\Gamma(v)$ in Bessel Function when

$r = r_1 = \sqrt{p^2} = v \neq Integer$

$$a_{2n} = \frac{(-1)^n}{2^{2n}\, n!(N+1)(N+2)\ldots(N+n)}\, a_0 \quad \rightarrow \quad \frac{(-1)^n}{2^{2n}\, n!(v+1)(v+2)\ldots(v+n)}\, a_0$$

$$a_0 = \frac{1}{2^N N!} \quad \rightarrow \quad \frac{1}{2^v \Gamma(v+1)}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n+N}\, n!(n+N)!} \quad \rightarrow \quad \frac{(-1)^n}{2^{2n+v}\, n!\Gamma(v+n+1)}$$

One of the Solutions of Bessel's Function (including $v = 0$):

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

is Bessel's Function of ***First Kind*** of order v :

$$J_v(x) = x^v \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+v}\, n!\, \Gamma(n+v+1)}$$