Finite Element Method (3): 2D FEM

Lecture 12-13

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Outline

- 2D using Triangular Elements
- 2D using Rectangular Elements

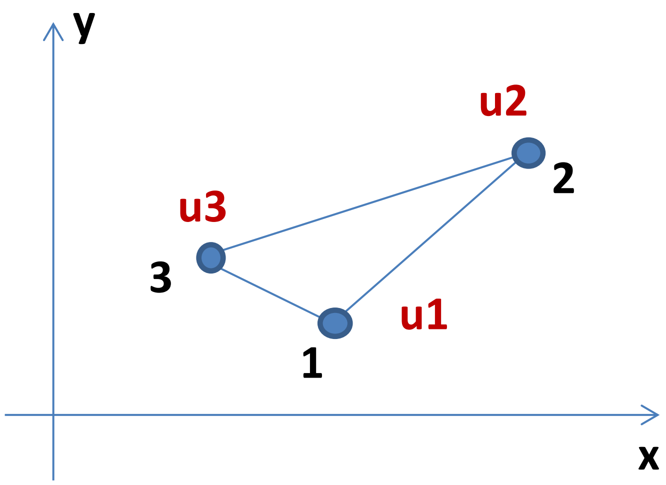
References

- S. Chapra and R. Canale, “Numerical Method’s for Engineers”, McGraw-Hill, 5th Ed., 2006
- S. Moaveni, “Finite Element Analysis, Theory and Application with Ansys”, Pearson Prentice Hall, 3rd Ed., 2008
- E. Thompson, “Introduction to the Finite Element Method: Theory, Programming, and Applications”, Wiley, 2005

Triangular Mesh

$$u(x,y) = a_0 + a_{1,1}x + a_{1,2}y$$

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_{1,1} \\ a_{1,2} \end{bmatrix} = \begin{bmatrix} u1 \\ u2 \\ u3 \end{bmatrix}$$



Find a_0, a_{11}, a_{12} (Cramer's Rule, LU, GE,...)

$$u = N_1u_1 + N_2u_2 + N_3u_3$$

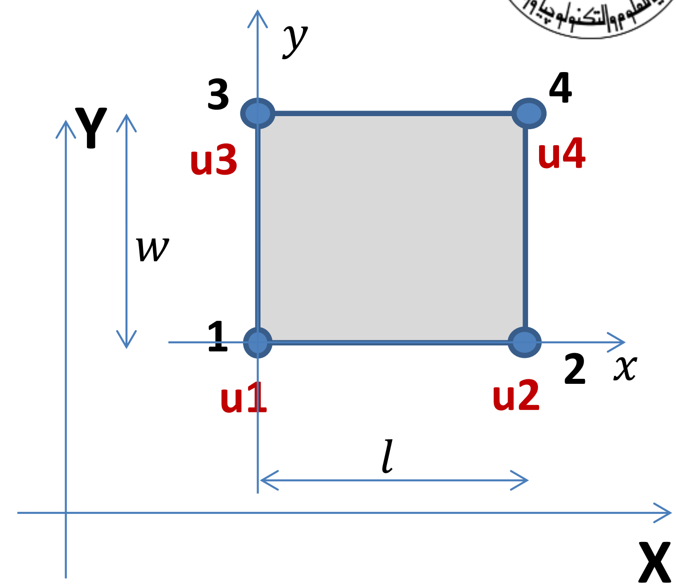
A_e=Area of triangular element=(1/2) Det(A)

- $N_1 = \frac{1}{2Ae}[x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$
- $N_2 = \frac{1}{2Ae}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$
- $N_3 = \frac{1}{2Ae}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$

Rectangular Mesh (Local Coordinates)

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$$

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & x_4 & x_4 y_4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} u1 \\ u2 \\ u3 \\ u4 \end{bmatrix}$$



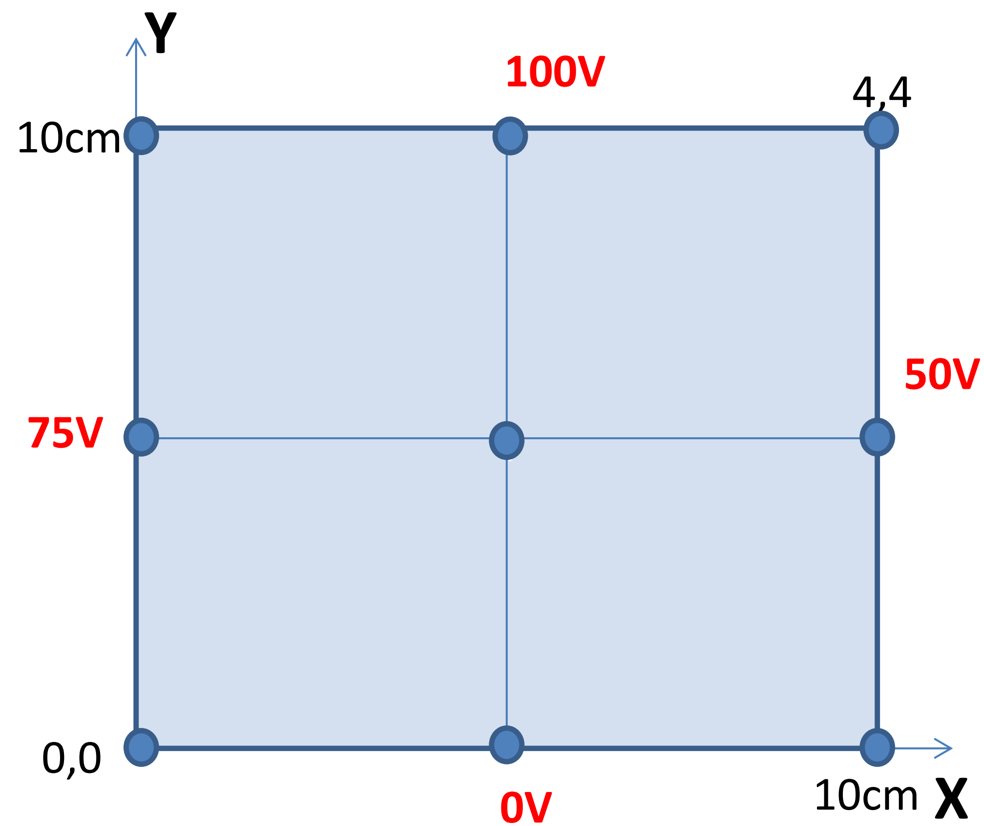
$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$N_1 = \left(1 - \frac{x}{l}\right)\left(1 - \frac{y}{w}\right),$$

$$N_2 = \frac{x}{l}\left(1 - \frac{y}{w}\right),$$

$$N_3 = \frac{y}{w}\left(1 - \frac{x}{l}\right), \qquad N_4 = \frac{xy}{lw}$$

Example: 2D Potential Equation with Drichlet Boundary Conditions



Example: 2D Potential Equation with Drichlet Boundary Conditions

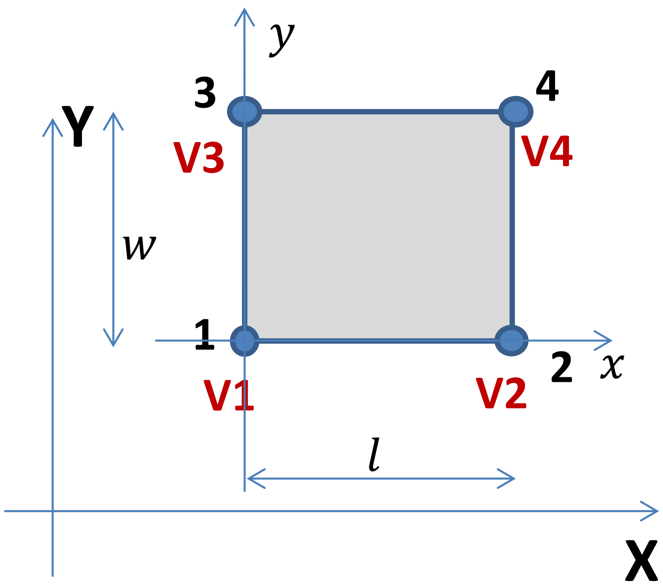
Element Equations:

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = -f(x)$$

$$f(x) = 0$$

$$\frac{\partial V}{\partial x} = -E_x$$

$$\frac{\partial V}{\partial y} = -E_y$$



Derivatives in 2D: $\frac{\partial V}{\partial x}$

$$V = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4 = [N]^T [V]$$

$$\frac{\partial V}{\partial x} = \frac{\partial [N]^T}{\partial x}[V] = \left\{\frac{\partial}{\partial x}[N_1 \quad N_2 \quad N_3 \quad N_4]\right\}[V]$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left[\left(1 - \frac{x}{l}\right)\left(1 - \frac{y}{w}\right)\right] = \left(1 - \frac{y}{w}\right)\left(\frac{-1}{l}\right) = \frac{y - w}{wl}$$

$$\frac{\partial N_2}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{l}\left(1 - \frac{y}{w}\right)\right] = \left(1 - \frac{y}{w}\right)\left(\frac{1}{l}\right) = \frac{w - y}{wl}$$

$$\frac{\partial N_3}{\partial x} = \frac{\partial}{\partial x}\left[\frac{y}{w}\left(1 - \frac{x}{l}\right)\right] = \frac{-y}{wl}, \qquad \frac{\partial N_4}{\partial x} = \frac{\partial}{\partial x}\left[\frac{xy}{lw}\right] = \frac{y}{wl}$$

$$\frac{\partial V}{\partial x} = \frac{1}{wl}[(y - w) \quad (w - y) \quad (-y) \quad (y)][V]$$

Derivatives in 2D: $\frac{\partial V}{\partial y}$

$$\frac{\partial N_1}{\partial y} = \frac{\partial}{\partial y}\left[\left(1-\frac{x}{l}\right)\left(1-\frac{y}{w}\right)\right] = \left(1-\frac{x}{l}\right)\left(\frac{-1}{w}\right) = \frac{x-l}{wl}$$

$$\frac{\partial N_2}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x}{l}\left(1-\frac{y}{w}\right)\right] = \frac{-x}{wl}$$

$$\frac{\partial N_3}{\partial y} = \frac{\partial}{\partial y}\left[\frac{y}{w}\left(1-\frac{x}{l}\right)\right] = \frac{l-x}{wl}, \qquad \frac{\partial N_4}{\partial x} = \frac{\partial}{\partial x}\left[\frac{xy}{lw}\right] = \frac{x}{wl}$$

$$\frac{\partial V}{\partial y} = \frac{1}{wl}\left[(x-l) \quad (l-x) \quad (-x) \quad (x)\right][V]$$

Residual Equations in 2D Rectangular F.E. Using Galerkin's Method

$$V = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4 = [N]^T [V]$$

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

$$R = \sum_{i=1}^{4} R_i = \sum_{i=1}^{4} \int_A N_i \left(\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} \right) dA$$

$$= \iint_A [N]^T \left(\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} \right) dxdy = 0$$

Where: $[N]^T = [N_1 \quad N_2 \quad N_3 \quad N_4]$

Green's Theory in 2D Finite Elements Using Galerkin's Method (2)

Use: $\frac{\partial}{\partial x}\left([N]^T \frac{\partial V}{\partial x}\right) = [N]^T \frac{\partial^2 V}{\partial x^2} + \frac{\partial [N]^T}{\partial x}\frac{\partial V}{\partial x}$

$$\rightarrow [N]^T \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left([N]^T \frac{\partial V}{\partial x}\right) - \frac{\partial [N]^T}{\partial x}\frac{\partial V}{\partial x}$$

Similarly:

$$[N]^T \frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left([N]^T \frac{\partial V}{\partial y}\right) - \frac{\partial [N]^T}{\partial y}\frac{\partial V}{\partial y}$$

By substituting:

$$R = I_1 + I_2$$

2D Rectangular Finite Elements Using Galerkin's Method (2)

The integral I_1 can be easily evaluated using derivatives of $[N]^T$

$$I_1 = \int_c^d \int_a^b \left(-\frac{\partial [N]^T}{\partial x} \frac{\partial V}{\partial x} - \frac{\partial [N]^T}{\partial y} \frac{\partial V}{\partial y} \right) dxdy$$

$$\frac{\partial [N]^T}{\partial x} = \frac{1}{wl} [(y - w) \quad (w - y) \quad (-y) \quad (y)][V]$$

$$\frac{\partial [N]^T}{\partial y} = \frac{1}{wl} [(x - l) \quad (l - x) \quad (-x) \quad (x)]$$

$$\frac{\partial V}{\partial x} = \frac{\partial [N]^T}{\partial x} [V], \qquad \frac{\partial V}{\partial y} = \frac{\partial [N]^T}{\partial y} [V]$$

2D Rectangular Finite Elements Using Galerkin's Method (3)

The integral I_2 can be evaluated using *Green's Theory* (Next Lecture):

$$I_2 = \int_A \left\{ \frac{\partial}{\partial x}\left([N]^T \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y}\left([N]^T \frac{\partial V}{\partial y} \right) \right\} dxdy$$