Integrals: Green's Theory

Lecture 14

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Outline

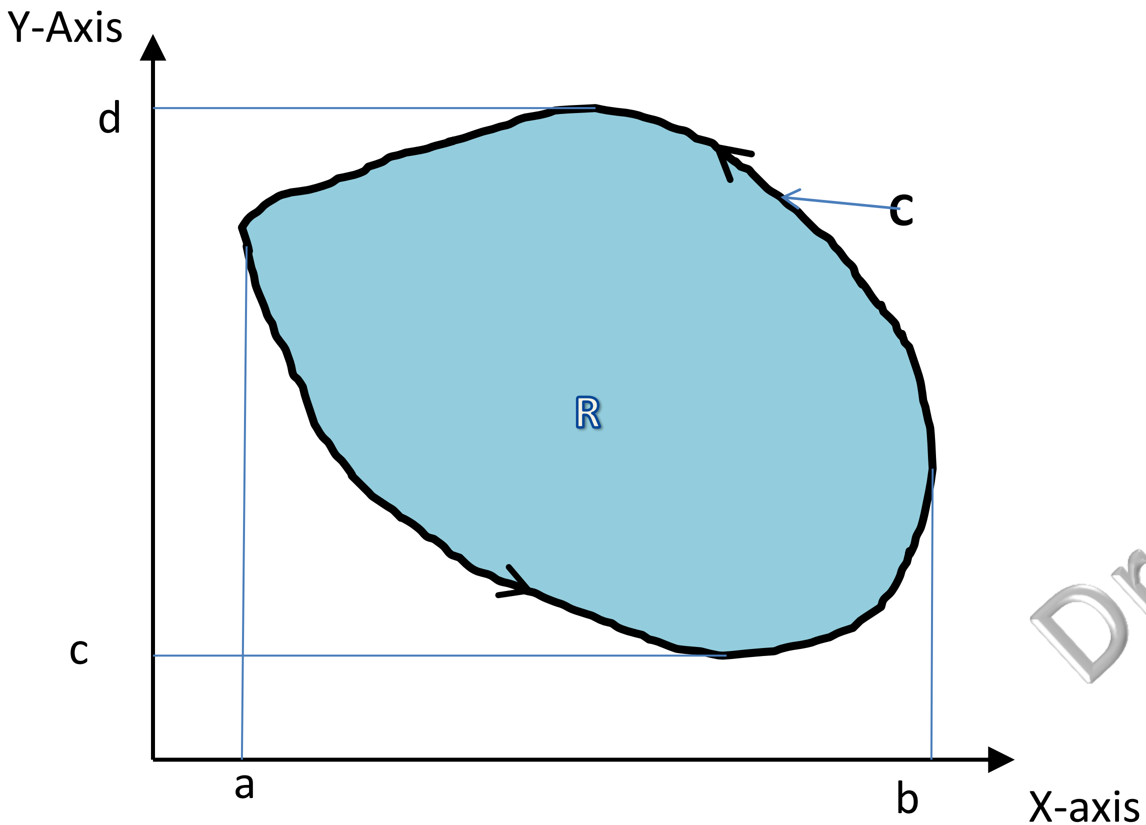
- Green's Theory on Surfaces
- Applications:
  - 2D Laplace Equation
  - 2D Finite Elements

References

- E. Kreyszig, “Advanced Engineering Mathematics”, 8th Ed., 1999
- S. Moaveni, “Finite Element Analysis, Theory and Application with Ansys”, Pearson Prentice Hall, 3rd Ed., 2008

Introduction

- Transforms ***Area Integral*** on an area (or Region "R") to a Line Integral around the contour ("C") surrounding this area

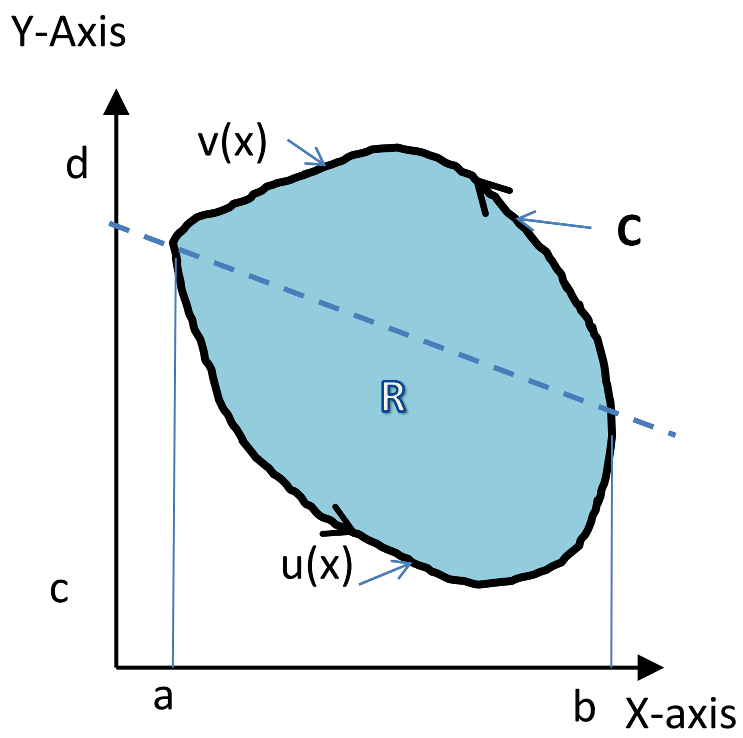


$$a \leq x \leq b$$

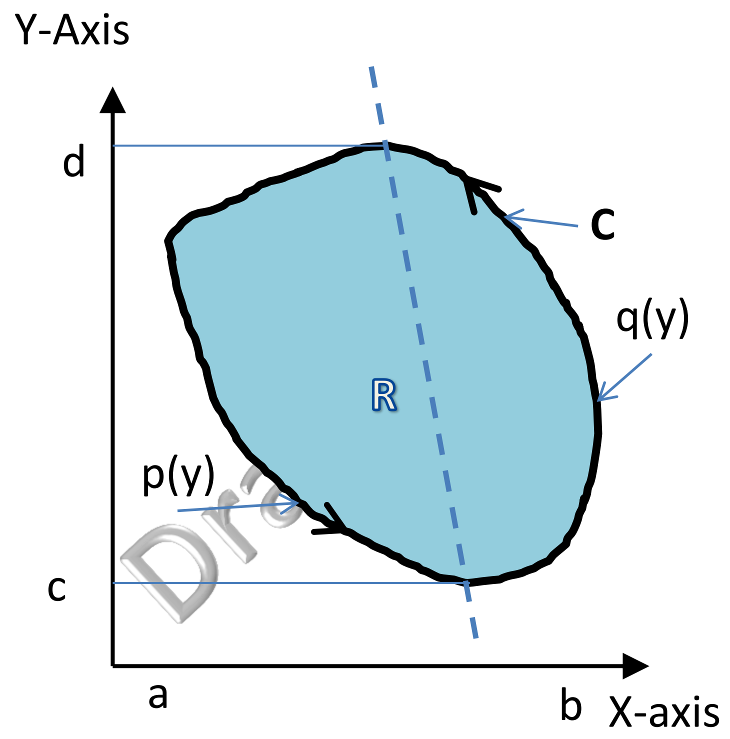
$$c \leq y \leq d$$

Definitions

$$a \le x \le b,$$
$$v(x) \le y \le u(x)$$



$$c \le y \le d,$$
$$p(y) \le x \le q(y)$$



Green's Theory

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \oint_C F_1 dx + F_2 dy$$

F_1 and F_2 are two functions which are continuous and have $\frac{\partial F_2}{\partial x}$ and $\frac{\partial F_1}{\partial y}$ continuous, within area "R"

Example of Green's Theory: Circle

Contour "C": $x^2 + y^2 = 1$

$(radius = \sqrt{1}, \quad area = \pi(\sqrt{1})^2 = \pi)$

$$F_1 = y^2 - 7y, \qquad F_2 = 2xy + 2x$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \iint_R [(2y+2) - (2y-7)]\, dx\, dy$$

$$= 9 \iint_R dx\, dy = 9\pi$$

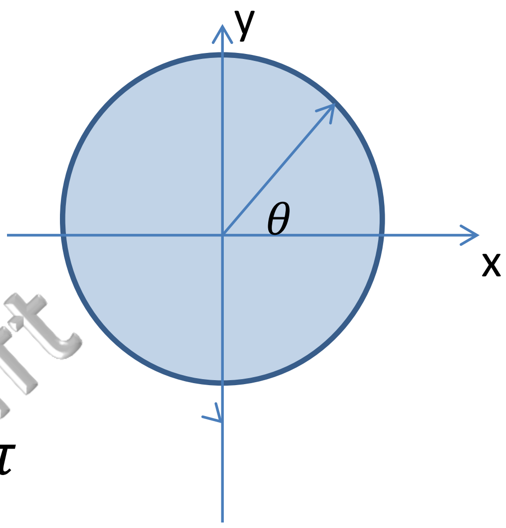
Example of Green's Theory: Circle

Contour "C": $x = \cos\theta\,,\;\; y = \sin\theta$

$$dx = -\sin\theta\;d\theta\,,\;\; dy = \cos\theta\;d\theta$$

$$F_1 = sin^2\theta - 7\sin\theta\,, \qquad F_2 = 2\cos\theta\sin\theta + 2\cos\theta$$

$$\oint_C F_1 dx + F_2 dy$$



$$= \int_0^{2\pi} (sin^2\theta - 7\sin\theta)(-\sin\theta)\;d\theta$$

$$+ \int_0^{2\pi} (2\cos\theta\sin\theta + 2\cos\theta)(\cos\theta)\;d\theta$$

$$= 0 + 7\pi + 0 + 2\pi = 9\,\pi$$

Proof of Green's Theory (1)

$$\iint_R \left(\frac{\partial F_1}{\partial y}\right) dx\,dy = \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy\right] dx$$

$$\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy = F_1(x, v(x)) - F_1(x, u(x))$$

$$\iint_R \left(\frac{\partial F_1}{\partial y}\right) dx\,dy = \int_a^b F_1(x, v(x))\,dx - \int_a^b F_1(x, u(x))\,dx$$

$$= -\int_b^a F_1\big(x, v(x)\big)\,dx - \int_a^b F_1\big(x, u(x)\big)\,dx = -\oint_C F_1(x, y)\,dx$$

Proof of Green's Theory (2)

$$\iint_R \left(\frac{\partial F_2}{\partial x}\right) dx\, dy = \int_c^d \left[\int_{p(y)}^{q(y)} \frac{\partial F_2}{\partial x} dx\right] dy$$

$$\int_{p(y)}^{q(y)} \frac{\partial F_2}{\partial x} dx = \int_{p(y)}^{q(y)} dF_2(x,y) = F_2(p(y),y) - F_2(q(y),y)$$

$$\iint_R \left(\frac{\partial F_2}{\partial x}\right) dx\, dy = \int_c^d F_2(p(y),y)\, dy - \int_c^d F_2(q(y),y)\ dy$$

$$= \int_c^d F_2(p(y),y)\, dy + \int_d^c F_2(q(y),y)\ dy = \oint_C F_2(x,y)\, dy$$

Proof of Green's Theory (3)

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial F_2}{\partial x}\right) dx\,dy - \iint_R \left(\frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \oint_C F_2(x,y)\,dy - [-\oint_C F_1(x,y)\,dx]$$

$$= \oint_C F_1 dx + F_2 dy$$

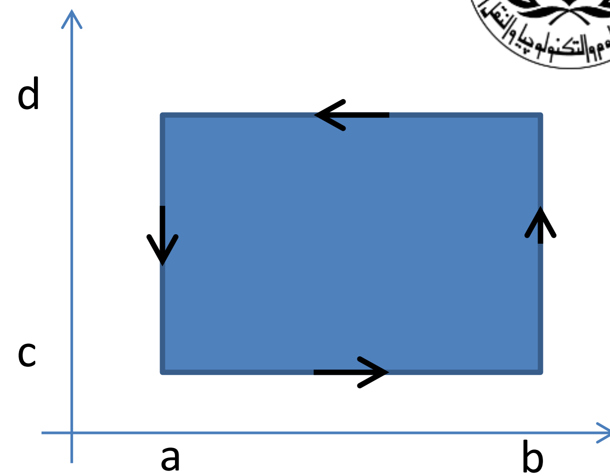
Applications for Green's Theory: **Laplace Equation**

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w$$

Let $F_1 = -\frac{\partial w}{\partial y}, F_2 = \frac{\partial w}{\partial x} \rightarrow \ \nabla^2 w = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

$$\iint_R [\nabla^2 w]\, dx\, dy = \oint \left\{\left[-\frac{\partial w}{\partial y}\right] dx + \left[\frac{\partial w}{\partial x}\right] dy\right\}$$

Applications for Green's Theory:
Laplace Equation (2)



Example: Laplace Eqn. within a 2D box

$w(x, y) = x^2 - y^2$

$F_1 = -\frac{\partial w}{\partial y} = 2y, F_2 = \frac{\partial w}{\partial x} = 2x,$

$\nabla^2 w = 2 - 2 = 0$

$$\iint_R [\nabla^2 w] dx\, dy = \oint_C \{[2y]dx + [2x]dy\}$$

$$I_1 = \oint_C 2y\, dx = \int_a^b 2c\, dx + 0 + \int_b^a 2d\, dx + 0 = 2c(b - a) + 2d(a - b)$$

$$I_2 = \oint_C 2x\, dy = \int_c^d 2b\, dy + 0 + \int_d^c 2a\, dx + 0 = 2b(d - c) + 2a(c - d)$$

$$\iint_R \nabla^2 w\, dx\, dy = I_1 + I_2 = 0$$

Green's Theory in 2D Finite Elements Using Galerkin's Method

From Lec. 13: The integral I_2 can be evaluated using Green's Theory:

$$I_2 = \int_A \left\{ \frac{\partial}{\partial x}\left([N]^T \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y}\left([N]^T \frac{\partial V}{\partial y} \right) \right\} dxdy$$

Let: $F_1 = -[N]^T \frac{\partial V}{\partial y}, \quad F_2 = [N]^T \frac{\partial V}{\partial x}$

$$\rightarrow I_2 = \iint_A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C F_1 dx + F_2 dy$$

$$= \oint_C \{-[N]^T \frac{\partial V}{\partial y} dx + [N]^T \frac{\partial V}{\partial x} dy\}$$

$$= \oint_C \{-[N]^T \frac{\partial [N]^T}{\partial y} [V]\, dx + [N]^T \frac{\partial [N]^T}{\partial x} [V]\, dy\}$$

Green's Theory in 2D Finite Elements Using Galerkin's Method (2)

From Lec. 13:

$$\frac{\partial V}{\partial x} = \frac{1}{wl}\left[(y-w) \quad (w-y) \quad (-y) \quad (y)\right][V]$$

$$\frac{\partial V}{\partial y} = \frac{1}{wl}\left[(x-l) \quad (l-x) \quad (-x) \quad (x)\right][V]$$