



Integrals (2): Gamma Function

Lecture 15

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References

- E. Kreyszig, “Advanced Engineering Mathematics”, 8th Ed., 1999
- M.R.Spiegel,” Fourier Analysis with Applications to Boundary Value Problems”, Schaum’s Outlines Series, McGraw-Hill
- R. Bronson, “ Differential in Equations”, 2nd Ed., Schaum’s Outlines Series, McGraw-Hill, 1994

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Outline

- Definition of Gamma Functions
- Properties of Gamma Functions
- Integrals using Gamma Functions
- Bessel Functions using Gamma Functions

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Definition of Gamma Function $\Gamma(v)$

$$\Gamma(v) = \int_0^{\infty} e^{-t} t^{v-1} dt$$

- $v \neq 0$ can be any positive or negative real or complex number
- Used to get factorials of non-integer numbers
- Evaluated using tables for small v

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Special Gamma Function $\Gamma(v)$

- $\Gamma(1) = \int_0^{\infty} e^{-t} t^0 dt = \int_0^{\infty} e^{-t} dt = -e^{-t} \Big|_0^{\infty} = 0 - (-1) = 1$
- $\Gamma(0) \rightarrow \text{undefined}$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

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Properties of Gamma Function

$$\Gamma(\nu + 1) = \int_0^{\infty} e^{-t} t^{\nu} dt = \int_0^{\infty} t^{\nu} d(-e^{-t})$$

Integrate by parts: $\Gamma(\nu + 1) = (-e^{-t}) t^{\nu} \Big|_0^{\infty} - \int_0^{\infty} (-e^{-t}) d(t^{\nu})$

$$= 0 + \nu \int_0^{\infty} e^{-t} t^{\nu-1} dt = \nu \Gamma(\nu)$$
$$\Gamma(\nu + 1) = \nu \Gamma(\nu)$$

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Properties of Gamma Function (2)

$$\Gamma(v+1) = v\Gamma(v) \rightarrow \Gamma(v) = \frac{\Gamma(v+1)}{v}$$
$$\rightarrow \Gamma(0) = \frac{\Gamma(0+1)}{0} \rightarrow \lim_{v \rightarrow 0^+} \frac{1}{0^+} = +\infty, \lim_{v \rightarrow 0^-} \frac{1}{0^-} = -\infty$$

For negative $x = v - 1, x \neq 0$:

$$\Gamma(x) = \Gamma(v-1) = \frac{\Gamma(v)}{v-1}$$
$$\rightarrow \Gamma(-1) = \Gamma(0-1) = \frac{\Gamma(0)}{0-1} = \frac{\pm\infty}{-1}$$
$$\rightarrow \Gamma\left(-\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}-1\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{1}{2}-1} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

Check for example: <https://www.physicsforums.com/threads/gamma-function-on-negative-fractions.489613/>



Generalization of Factorials using Gamma Function

$$\Gamma(2) = 1 \quad \Gamma(1) = 1!, \quad \Gamma(3) = 2\Gamma(2) = 2!$$

For n =integer: $\Gamma(n + 1) = n!$

For any $x = n + v$, where $v > 0$ is not an integer:

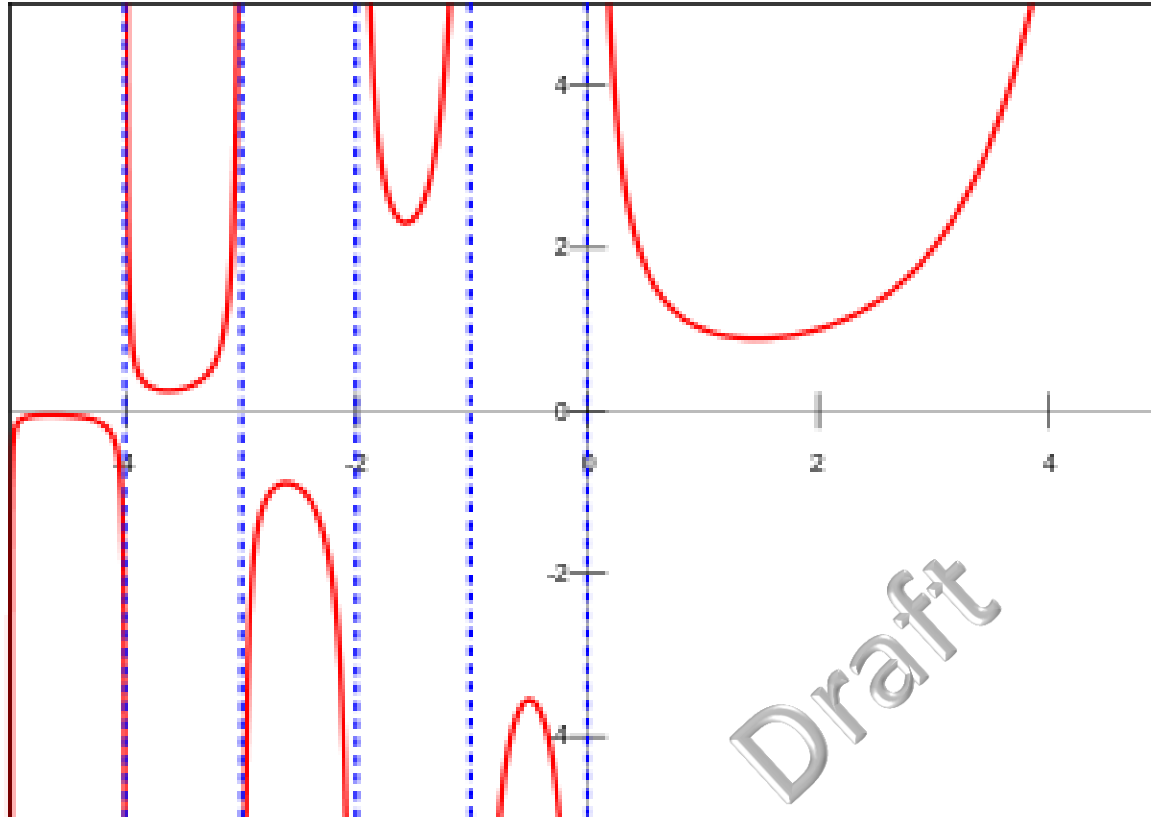
$$\begin{aligned}\Gamma(x) &= \Gamma(v + n) = (v + n - 1)\Gamma(v + n - 1) \\ &= (v + n - 1)(v + n - 2)\Gamma(v + n - 2) \\ \Gamma(v + n) &= (v + n - 1)(v + n - 2) \dots (v + 1)\Gamma(v + 1)\end{aligned}$$

Example: $\Gamma(4.5) = (3.5)(2.5)(1.5)(0.5)\Gamma(0.5) = 5.5625\sqrt{\pi}$



Plots of $\Gamma(v)$

Gamma function



Source: http://en.wikipedia.org/wiki/Gamma_function



Example Integrals Using $\Gamma(v)$

- **Example:** Integral used to calculate number of conduction electrons in semiconductors, for Energy $E = E_c$ to $E \rightarrow \infty$:

$$I_1 = \int_{E-E_c=0}^{\infty} \sqrt{E - E_c} e^{-(E-E_c)} d(E - E_c)$$

Using $x = E - E_c$

$$\begin{aligned} \rightarrow I_1 &= \int_0^{\infty} x^{1/2} e^{-x} dx = \int_0^{\infty} e^{-x} x^{\left(\frac{3}{2}-1\right)} dx = \Gamma\left(\frac{3}{2}\right) \\ &= \Gamma(1 + 0.5) = 0.5 \Gamma(0.5) = \sqrt{\pi}/2 \end{aligned}$$



Reminder: **Bessel's Function of First Kind** for $r = r_1 = p = N = \text{Integer}$

General Formula ($N = p = 0, 1, 2, \dots$ i. e. $p^2 = 0, 1, 4, 9, \dots$):

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(N+1)(N+2)\dots(N+n)} a_0$$

a_0 is an arbitrary coeff., typically expressed as: $a_0 = \frac{1}{2^N N!}$

$$a_{2n} = \frac{(-1)^n}{2^{2n+N} n! (n + N)!}$$

One of the solutions of Bessel's Equation is **Bessel's Function of First Kind** of order N :

$$J_N(x) = x^N \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+N} n! (N + n)!}$$

Where $N = p = \text{Integer}$



Application to $\Gamma(v)$ in Bessel Function when

$$r = r_1 = \sqrt{p^2} = v \neq \text{Integer}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(N+1)(N+2)\dots(N+n)} a_0 \rightarrow \frac{(-1)^n}{2^{2n} n!(v+1)(v+2)\dots(v+n)} a_0$$

$$a_0 = \frac{1}{2^N N!} \rightarrow \frac{1}{2^v \Gamma(v+1)}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n+N} n!(n+N)!} \rightarrow \frac{(-1)^n}{2^{2n+v} n! \Gamma(v+n+1)}$$

One of the Solutions of Bessel's Function (including $v = 0$):

$$x^2 y'' + x y' + (x^2 - v^2) y = 0$$

is Bessel's Function of First Kind of order v :

$$J_v(x) = x^v \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+v} n! \Gamma(n+v+1)}$$