Special Functions

Lecture 8

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Outline

- Singular Points
- The Method of Frobenius
- Bessel Function
- Gamma Function
- Legendre Polynomial

References

- G. F. Simmons and S. G. Krantz, “Differential Equations: Theory, Technique, and Practice”, McGraw-Hill, 2007
- D.L. Powers, “Boundary Value Problems and Partial Differential Equations”, Academic Press, 6th Ed., 2010
- R. Bronson, “ Differential in Equations”, 2nd Ed., Schaum’s Outlines Series, McGraw-Hill, 1994
- E. Kreyszig, “Advanced Engineering Mathematics”, John Wiley & Sons

Frobenius Method

- Find a solution for the 2nd order differential equation:

$$y'' + \frac{P(x)}{x} y' + \frac{Q(x)}{x^2} y = 0$$

Multiplying by x^2 : $\quad x^2 y'' + xP(x)y' + Q(x)y = 0$

Frobenius Method: A series solution can be found near $x_o = 0$ as:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

If Taylor's expansion $\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$ at $x_o = 0$ is Analytical for both:

$P(x)$ and $Q(x)$

-> Series Solution of D.E. can be used

x_o is called *Regular Singular Point*

If singularity is around $x_o \neq 0$, *make a translation of axis to* $(x - xo)$

Frobenius Method: **Procedure**

$$x^2y'' + P(x)xy' + Q(x)y = 0$$

Frobenius Method: $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$

$$y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \cdots + a_2x^{r+n} + \cdots$$

$$y' = ra_0x^{r-1} + (r+1)a_1x^r + (r+2)a_2x^{r+1} + \cdots$$
$$+(r+n)a_nx^{r+n-1} + \cdots$$

$$y'' = (r-1)ra_0x^{r-2} + (r+1)ra_1x^{r-1} + (r+2)(r+1)a_2x^r$$
$$+ \cdots + (r+n+2)(r+n+1)a_nx^{r+n} + \cdots$$

- Substitute in DE and Equate coefficients for x. This results in:
  – Indicial Equation by equating coefficients of $\boldsymbol{x^0}$: use to find two values of $\boldsymbol{r}$
  – Recurrence Formula by equating coefficients of $\boldsymbol{x^n}$

Frobenius Method: **Indicial Equation**

- A quadratic equation in r
- Used to find r_1, r_2
- Results in two solutions $y_{r1}(x)$ and $y_{r2}(x)$

Example Application of Frobenious Method:
Bessel's Equation

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - p^2}{x^2}\right) y = 0 \rightarrow x^2 y'' + x y' + (x^2 - p^2) y = 0$$

- $P(x)/x$ and $Q(x)/x^2$ are singular at x=0
- $P(x)$ and $Q(x)$ are analytical at x=0

Example Applications of Bessel's Function:

- Laplace Eq. in cylindrical coordinates
- Schrodinger's equation in radial form

Application: Bessel's Equation
Laplace Eq. in Cylindrical Cooridnates,

$$y'' + \frac{1}{x}y' + \left(\frac{x^2-p^2}{x^2}\right)y = 0 \rightarrow x^2y'' + xy' + (x^2 - p^2)y = 0$$

- Using Frobenius Method for series representation of y, and substituting in DE, and equating coefficients:

n=0: $r(r-1)a_0 + ra_0 - p^2a_0 = 0$ (Indicial Equation)

n=1: $(r+1)ra_1 + (r+1)a_1 - p^2a_1 = 0$

$\rightarrow [(r+1)^2 - p^2]a_1 = 0 \rightarrow a_1 = 0$ *(since $r+1=p$ cannot be imposed)*

n=s: $(s+r)(s+r-1)a_s + (s+r)a_s - a_{s-2} - p^2a_s = 0$

Indicial Equation: $(r+p)(r-p) = 0 \rightarrow r_1 = p \geq 0,\ r_2 = -p$

Recurrence Formula: $(s+r+p)(s+r-p) - a_{s-2} = 0$

Bessel's Equation: **Recurrence Formula**

$r = r_1 = p$

Recurrence Formula:

$$(s+r+p)(s+r-p)a_s - a_{s-2} = 0$$

For $r = p$: $$(s+2p)sas + a_{s-2} = 0$$

$a_1 = 0 \rightarrow a_3 = 0, a_5 = 0 \quad \rightarrow$ use only even indices: $s \Rightarrow 2n$

$$(2n+2p)(2n)\ a_{2n} + a_{2n-2} = 0$$

Recurrence Formula: $a_{2n} = \dfrac{-1}{2^2 n(p+n)} a_{2n-2}$

General Formula, in terms of a_0 :

$$a_{2n} = \frac{(-1)^n}{2^{2n}\, n!\,(p+1)(p+2)\ldots(p+n)} a_0$$

Bessel's Equation: **Bessel's Functon of First Kind for $r = r_1 = p = N = Integer$**

General Formula ($N = p = 0,1,2,\ldots$ $i.e.\, p^2 = 0,1,4,9,\ldots$):

$$a_{2n} = \frac{(-1)^n}{2^{2n}\, n!(N+1)(N+2)\ldots(N+n)}\, a_0$$

a_0 is an arbitrary coeff., typically expressed as: $a_0 = \frac{1}{2^N N!}$

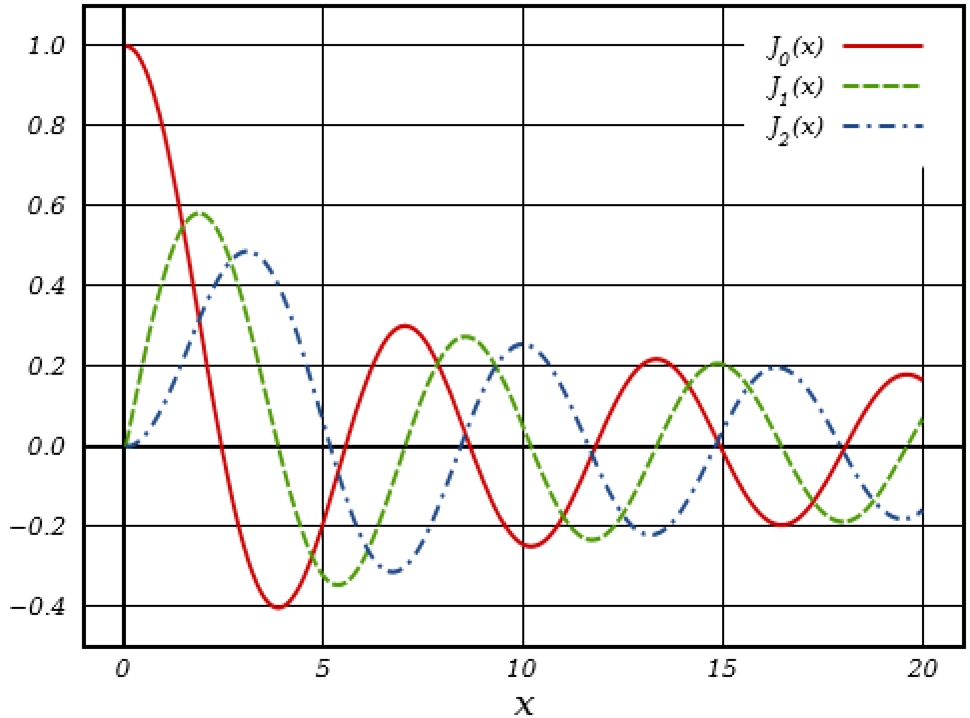
$$a_{2n} = \frac{(-1)^n}{2^{2n+N}\, n!\,(n+N)!}$$

One of the solutions of Bessel's Equation is ***Bessel's Function*** of ***First Kind*** of order ***N***:

$$J_N(x) = x^N \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+N}\, n!\,(N+n)!}$$

Where $N = p = Integer$

Plots of $J_N(x)$



Reprinted from Wolfram Mathworld:
http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

Notes

- $J_N(x)$ converges at x=0
- Solution exists for $p \neq Integer$ using Gamma Functions
- Another solution exists Y(x) which does not converge at x=0