Gaussian Elimination and Matrix Factorization

Lecture 1

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Outline

- Definitions
- Gaussian Elimination for solution of a system of linear equations
- Lower-Upper Factorization
- Examples

Definitions

$$A\mathbf{x} = \mathbf{r}$$

A: n x n Matrix

x : n x 1 vector of unknowns

r: n x 1 vector of “given” right hand side (RHS) values

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Problem: Find $x_1, \ldots, x_n$

Gaussian Elimination (1)

I. Forward Substitution

Eliminate x_1 from first equation:

$$x_1 = -\frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3 + \frac{1}{a_{11}} r_1$$

This is equivalent to:

$$x_1 + \frac{a_{12}}{a_{11}} x_2 + \frac{a_{13}}{a_{11}} x_3 = \frac{1}{a_{11}} r_1$$

Substitute x_1 in next 2 equations (first column in matrix):

$$\left(a_{22} - \frac{a_{21}\, a_{12}}{a_{11}}\right) x_2 + \left(a_{23} - \frac{a_{21} a_{13}}{a_{11}}\right) x_3 = r_2 - \frac{a_{21}}{a_{11}} r_1$$

$$\left(a_{32} - \frac{a_{31} a_{12}}{a_{11}}\right) x_2 + \left(a_{33} - \frac{a_{31}\, a_{13}}{a_{11}}\right) x_3 = r_3 - \frac{a_{31}}{a_{11}} r_1$$

This is equivalent to:

$$x_1 + b_{12} x_2 + b_{13} x_3 = s_1$$
$$0 + b_{22} x_2 + b_{33} x_3 = s_2$$
$$0 + b_{32}\, x_2 + b_{33} x_3 = s_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{11} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{12}}{a_{11}} \\ 0 & a_{22} - \frac{a_{21}\, a_{12}}{a_{11}} & a_{23} - \frac{a_{21} a_{13}}{a_{11}} \\ 0 & a_{32} - \frac{a_{31} a_{12}}{a_{11}} & a_{33} - \frac{a_{31} a_{13}}{a_{11}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{a_{11}} r_1 \\ r_2 - \frac{a_{21}}{a_{11}} r_1 \\ r_3 - \frac{a_{31}}{a_{11}} r_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Gaussian Elimination (2)

Next, eliminate x_2:

$$x_1 + b_{12}x_2 + b_{13}x_3 = s_1$$

$$0 + x_2 + \frac{b_{23}}{b_{22}} x_3 = \frac{s_2}{b_{22}}$$

$$0 + 0 + (b_{33} - b_{32}\frac{b_{23}}{b_{22}})x_3 = s_3 - \frac{b_{32}}{b_{22}} s_2$$

$$\begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & \frac{b_{23}}{b_{22}} \\ 0 & b_{32} & b_{33} - b_{32}\frac{b_{23}}{b_{22}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1 \\ \frac{s_2}{b_{22}} \\ s_3 - \frac{b_{32}}{b_{22}} s_2 \end{bmatrix}$$

Note that this is the same as :

$$x_2 = \frac{1}{a_{22} - \frac{a_{21}a_{12}}{a_{11}}}\left[-\left(a_{23} - \frac{a_{21}a_{13}}{a_{11}}\right)x_3 + r_2 - \frac{a_{21}}{a_{11}} r_1\right]$$

Now, rewrite as:

$$x_1 + b_{12}x_2 + b_{13}x_3 = s_1$$

$$0 + x_2 + c_{23}x_3 = k_2$$

$$0 + 0 + c_{33}\, x_3 = k_3 \rightarrow x_3 = k_3/c_{33}$$

$$\begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s1 \\ k_2 \\ \frac{k_3}{c_{33}} \end{bmatrix}$$

II. Backwards Substitution

Solve previous equations in reverse order

- Find x_2 using x_3
- Find x_1 using x_2 and x_3

Example for Gaussian Elimination

I. Forward Substitution

$$\begin{bmatrix}2 & 3 & 1\\ 2 & 5 & 4\\ 8 & 7 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1\\ -1\\ 2\end{bmatrix}$$

$$x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3 = \frac{1}{a_{11}}r_1$$

$$x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 = \frac{1}{2}\ 1$$

$$\begin{bmatrix}1 & \frac{3}{2} & \frac{1}{2}\\ 2 & 5 & 4\\ 8 & 7 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ -1\\ 2\end{bmatrix}$$

Substitute x_1 in next 2 equations (first column in matrix):

$$\left(a_{22} - \frac{a_{21}\,a_{12}}{a_{11}}\right) x_2 + \left(a_{23} - \frac{a_{21}a_{13}}{a_{11}}\right)x_3 = r_2 - \frac{a_{21}}{a_{11}}r_1$$

$$\left(a_{32} - \frac{a_{31}a_{12}}{a_{11}}\right) x_2 + \left(a_{33} - \frac{a_{31}a_{13}}{a_{11}}\right)x_3 = r_3 - \frac{a_{31}}{a_{11}}r_1$$

$$\begin{bmatrix}1 & \frac{3}{2} & \frac{1}{2}\\ 0 & 5-\frac{2\,.\,3}{2} & 4-\frac{2\,.\,1}{2}\\ 0 & 7-\frac{8\,.\,3}{2} & 6-\frac{8\,.\,1}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ -1-\frac{2}{2}\,1\\ 2-\frac{8}{2}\,1\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{3}{2} & \frac{1}{2}\\ 0 & 2 & 3\\ 0 & -5 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ -2\\ -2\end{bmatrix}$$

Next, eliminate x_2:

$$x_1 + b_{12}x_2 + b_{13}x_3 = s_1$$

$$0 + x_2 + \frac{b_{23}}{b_{22}}x_3 = \frac{s_2}{b_{22}}$$

$$0 + 0 + \left(b_{33} - b_{32}\frac{b_{23}}{b_{22}}\right)x_3 = s_3 - \frac{b_{32}}{b_{22}}s_2$$

$$\begin{bmatrix}1 & \frac{3}{2} & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 2-\frac{-5\,.3}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ -1\\ -2-\frac{5(-2)}{2}\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{3}{2} & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ -1\\ -\frac{14}{19}\end{bmatrix} \rightarrow \boldsymbol{x_3 = -\frac{14}{19}}$$

Example for Gaussian Elimination (2)

II. Backwards Substitution

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ -\frac{14}{19} \end{bmatrix}$$

$$x_3 = -\frac{14}{19}$$

Solve previous equations in reverse order

- Find x_2 using x_3

$$x_2 = -\frac{3}{2}x_3 - 1 = -\frac{3}{2}\left(-\frac{14}{19}\right) - 1 = \frac{2}{19}$$

- Find x_1 using x_2 and x_3

$$x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}\ 1$$

$$= -\frac{3}{2}\left(\frac{2}{19}\right) - \frac{1}{2}\left(-\frac{14}{19}\right) + \frac{1}{2}\ 1 = \frac{27}{38}$$

Lower-Upper (LU) Factorization

Systematic way of implementing Gaussian Elimination

For the linear system of equations: $A\boldsymbol{x} = \boldsymbol{r}$

Assume we can write the nxn coefficient matrix A as a product of two nxn matrices L and U

$$A = LU$$

Then $A\boldsymbol{x} = \boldsymbol{r}$ becomes: $LU\boldsymbol{x} = \boldsymbol{r}$

Next, introduce the intermediate vector v: $\boldsymbol{v} = U\boldsymbol{x}$

Then the equations become: $L\boldsymbol{v} = \boldsymbol{r}$

$\boldsymbol{v}$ can be solved using Gaussian Elimination of: $L\boldsymbol{v} = \boldsymbol{r}$

Then $U\boldsymbol{x} = \boldsymbol{v}$ can also be solved to get $\boldsymbol{x}$ vector

Lower-Upper (LU) Factorization

Assume than both L and U can be written in triangular form such that:

L has all elements above diagonal equal to zero

U has all elements under diagonal equal to zero, and all diagonal entries equal to 1's:

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ \vdots & \ddots & 0 \\ l_{n1} & \cdots & l_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & \cdots & u_{1n} \\ 0 & 1 & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

The equations $\boldsymbol{Lv = r}$ Can be solved using ***Forward*** Gaussian Elimination only!

The equations $\boldsymbol{Ux = v}$ can be solved after that using ***Backwards*** Gaussian Elimination only!

Calculation of L and U

The product of LU is

$$LU = \begin{bmatrix} l_{11} & 0 & 0 \\ \vdots & \ddots & 0 \\ l_{n1} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & \cdots & u_{1n} \\ 0 & 1 & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{1k} & \dots a_{1n} \\ \vdots & \ddots a_{jk} & a_{jn} \\ a_{n1} & \cdots & \dots a_{nn} \vdots \end{bmatrix} = A$$

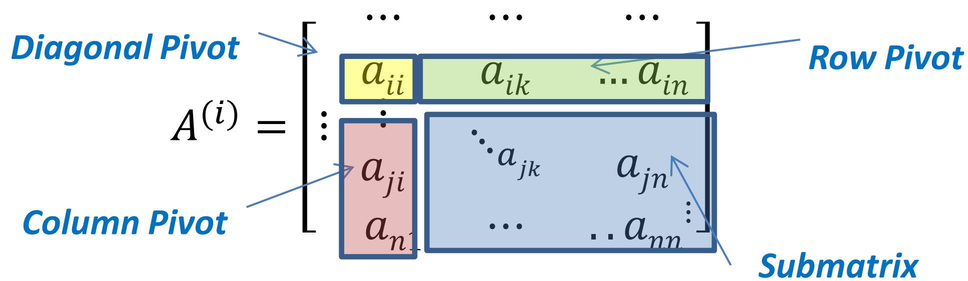
The equations $\boldsymbol{Lv = r}$ Can be solved using ***Forward*** Gaussian Elimination only!

The equations $\boldsymbol{Ux = v}$ can be solved after that using ***Backwards*** Gaussian Elimination only!

Calculation of L and U

To find elements in the LU matrix:

$$A^{(0)} \rightarrow A^{(1)} \ldots \rightarrow A^{(i)} \ldots \rightarrow A^{(n-1)} \rightarrow \begin{bmatrix} l_{11} & u_{1k} & \ldots u_{1n} \\ \vdots & \ddots l_{jk} & u_{jn} \\ l_{n1} & \cdots & ..l_{nn} \vdots \end{bmatrix}$$



For i=Outer loop number (i=1 to n-1) :

For j=row number, k=column number

Pivot Elements:

Diagonal Pivots (no change): $a_{ii}{}^{(i)} = a_{ii}{}^{(i-1)}$

Column Pivots (no change): $a_{kj}{}^{(i)} = a_{kj}{}^{(i-1)}$

Row Pivots: Divide by diagonal pivot: $a_{ij}{}^{(i)} = \dfrac{a_{ij}{}^{(i-1)}}{a_{ii}{}^{(i)}}$

Submatrix Elements: $a_{jk}{}^{(i)} = a_{jk}{}^{(i-1)} - \dfrac{a_{ik}{}^{(i-1)} a_{ji}{}^{(i-1)}}{a_{ii}{}^{(i-1)}}$

Example for LU Factorization: **Calculate L & U**

(Same as Gaussian Elimination example)

Initially:

$$A^{(0)} = A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 5 & 4 \\ 8 & 7 & 6 \end{bmatrix}$$

For i=1

$$A^{(1)} = \begin{bmatrix} 2 & 3/2 & 1/2 \\ 2 & 5 - \frac{3\,.\,2}{2} & 4 - \frac{1\,.\,2}{2} \\ 8 & 7 - \frac{3\,.\,8}{2} & 6 - \frac{1\,.\,8}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3/2 & 1/2 \\ 2 & 2 & 3 \\ 8 & -5 & 2 \end{bmatrix}$$

For i=2: $A^{(2)} = \begin{bmatrix} 2 & 3/2 & 1/2 \\ 2 & 2 & 3/2 \\ 8 & -5 & 2 - \frac{3\,(-5)}{2} \end{bmatrix}$

$$= \begin{bmatrix} 2 & 3/2 & 1/2 \\ 2 & 2 & 3/2 \\ 8 & -5 & 19/2 \end{bmatrix}$$

Now we have L & U:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 8 & -5 & 19/2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Example for LU Factorization: **Forward and Backwards Substitution**

Solve using Gaussian Elimination (Forward Substitution):

$$L\boldsymbol{v} = \boldsymbol{r}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 8 & -5 & 19/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$v_1 = \frac{1}{2}$$

$$2\left(\frac{1}{2}\right) + 2v_2 = -1$$

$$\rightarrow v_2 = \frac{1}{2}(-1 - 1) = -1$$

$$8\left(\frac{1}{2}\right) - 5(-1) + \frac{19}{2}v_3 = 2$$

$$\rightarrow v_3 = \frac{2}{19}(2 - 9) = -\frac{14}{19}$$

Solve using Gaussian Elimination (Backwards Substitution):

$$U\boldsymbol{x} = \boldsymbol{v}$$

$$\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ -14/19 \end{bmatrix}$$

$$x_3 = -\frac{14}{19} \qquad (Note:\ x_3 = v_3)$$

$$(1)x_2 + \frac{3}{2}\left(-\frac{14}{19}\right) = -1$$

$$\rightarrow x_2 = \frac{1}{1}\left(-1 + \frac{21}{19}\right) = \frac{2}{19}$$

$$(1)x_1 + \frac{3}{2}\left(\frac{2}{19}\right) + \frac{1}{2}\left(-\frac{14}{19}\right) = \frac{1}{2}$$

$$x_1 = \frac{1}{1}\left(\frac{1}{2} - \frac{3}{19} + \frac{7}{19}\right) = \frac{27}{38}$$

Same as Gaussian Elimination

Some Properties of LU Matrix

- If only the right hand side vector is changing (i.e. fixed A matrix), then the LU matrix needs to be calculated *only once*:
  - *Only make the forward and backwards substitution steps for each new RHS vector*
- If any of the row pivots elements is zero, then all the elements in the corresponding column in the new submatrix are unchanged
  - *No need to calculate*
- If any of the column pivots elements is zero, then all the elements in the corresponding row in the new submatrix are unchanged
  - *No need to calculate*
- *Remember that LU factorization is the practical method which replaces matrix inversion:*

$$Ax = r \rightarrow A^{-1}Ax = x = A^{-1}r$$