DISCRETE –TIME FOURIER TRANSFORM (DTFT)

The **discrete-time Fourier Transform**

$$x\left(e^{jw}\right) = \Im\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

The **Inverse discrete-time Fourier Transform (IDTFT)**

$$x(n) = \Im^{-1}\left[X\left(e^{jw}\right)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{jw}\right).e^{jwn}dw$$

Notes:

- ***$X(e^{jw})$*** is a **complex valued continuous** function
- **w** = 2π **f [rad/sec]**
- **f is the digital frequency measured in [C/S]**

EXAMPLE 3.1

Consider the signal $x(n)=(0.5)^n U(n)$

a- **Determine** its **DTFT.**
b- **Evaluate** $X(e^{jw})$ at **w** = 0, 0.25**π**, 0.5**π** , 0.75 **π** and **π**
c- **Plot** its **magnitude** , **angle**, **real part** and **imaginary part**

Solution:

a- Since the **signal** ***x(n)*** is **absolutely summable**, therefore, **DTFT exists**:

$$X(e^{jw})=\sum_{n=-\infty}^{\infty} x(n)e^{-jwn}=\sum_{0}^{\infty}(0.5)^n e^{-jwn}$$

$$\sum_{0}^{\infty}\left(0.5\times e^{-jw}\right)^n=\frac{1}{1-\left(0.5\times e^{-jw}\right)}$$

$$X(e^{jw})=\frac{e^{jw}}{e^{jw}-0.5}=\frac{\cos(w)+j\sin(w)}{(\cos(w)+j\sin(w))-(0.5)}=\frac{\cos(w)+j\sin(w)}{(\cos(w)-(0.5))+j\sin(w)}$$

$$X\left(e^{jw}\right)=\frac{\left(\cos(w)+j\sin(w)\right)\times\left(\left(\cos(w)-(0.5)\right)-j\sin(w)\right)}{\left(\cos(w)-0.5\right)^2+\left(\sin(w)\right)^2}$$

$$X\left(e^{jw}\right)=\frac{\left[\left(\cos(w)\right)^2-(0.5)\cos(w)+\left(\sin(w)\right)^2\right]+j\left[\sin(w)\cos(w)-\sin(w)\cos(w)-0.5\sin(w)\right]}{\left(\cos(w)\right)^2+\left(\sin(w)\right)^2+0.25-\cos(w)}$$

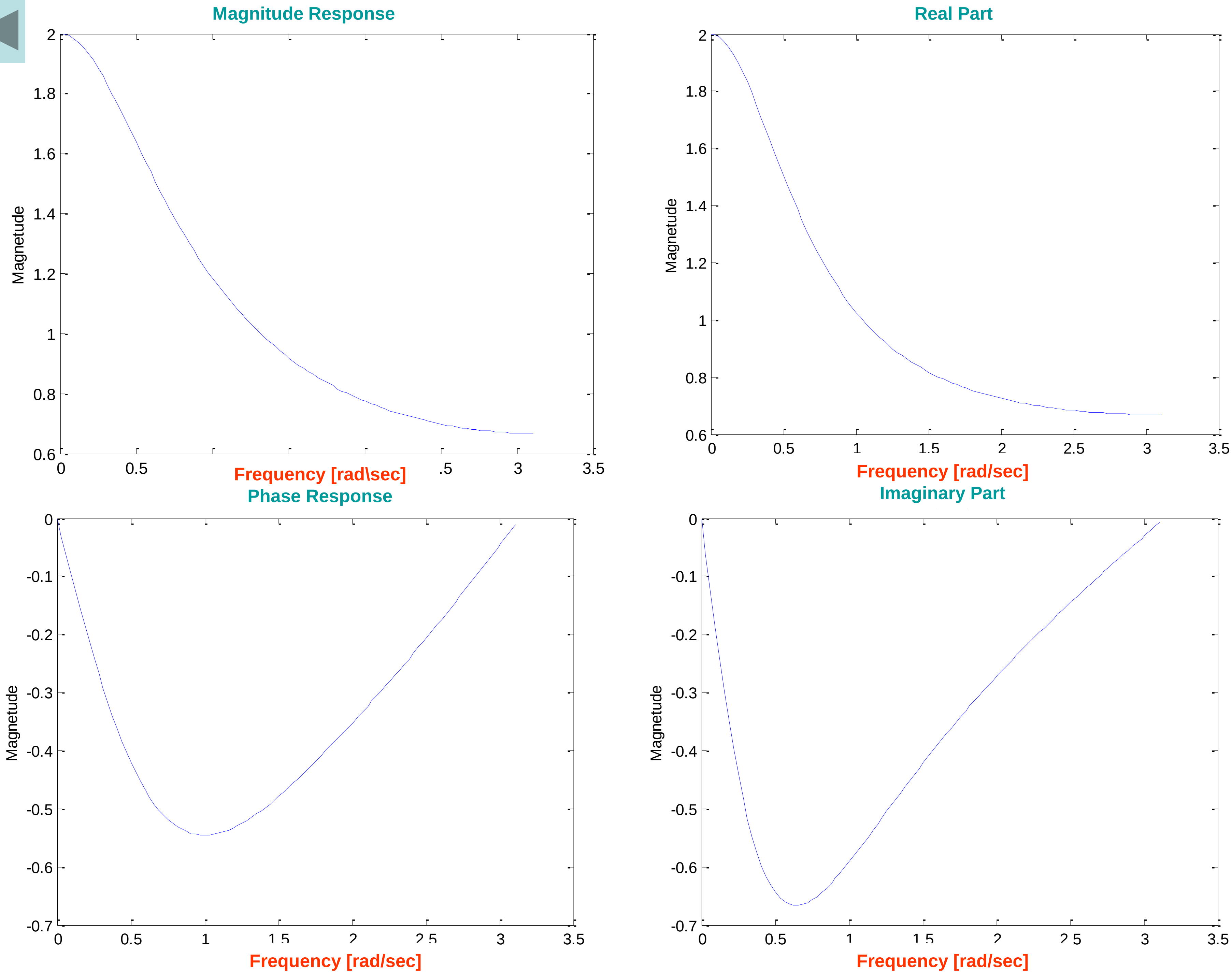
$$X\left(e^{jw}\right)=\frac{\left[1-0.5\cos(w)\right]-j0.5\sin(w)}{1.25-\cos(w)}$$

$$\left|X\left(e^{jw}\right)\right|=\sqrt{\frac{\left(1-0.5\cos(w)\right)^2+\left(0.5\sin(w)\right)^2}{\left(1.25-\cos(w)\right)^2}}$$

$$\operatorname{Re}\left[X\left(e^{jw}\right)\right]=\frac{1-0.5\cos(w)}{1.25-\cos(w)}$$

$$X\left(e^{jw}\right)=\tan^{-1}\frac{-0.5\sin(w)}{1-0.5\cos(w)}$$

$$\operatorname{Im}\left[X\left(e^{jw}\right)\right]=\frac{-0.5\sin(w)}{1.25-\cos(w)}$$


Magnitude Response
Magnetude
Frequency [rad\sec]
Real Part
Magnetude
Frequency [rad/sec]
Phase Response
Magnetude
Frequency [rad/sec]
Imaginary Part
Magnetude
Frequency [rad/sec]

EXAMPLE 3.2

For the system described by its impulse response $h(n) = (0.9)^n U(n)$

Do the following

a- Determine the **frequency response**

b- Plot the **magnitude and phase responses**

Solution

a- Using DTFT we find :

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h(n) e^{-jwn} = \sum_{n=0}^{\infty} (0.9)^n e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} \left(0.9e^{-jw}\right)^n = \frac{1}{1-(0.9)e^{-jw}} = \frac{1}{1-(0.9)\cos(w) + j(0.9)\sin(w)}$$

Hence

$$\left|H(e^{jw})\right| = \sqrt{\frac{1}{(1-0.9\cos(w))^2 + (0.9\sin(w))^2}}$$

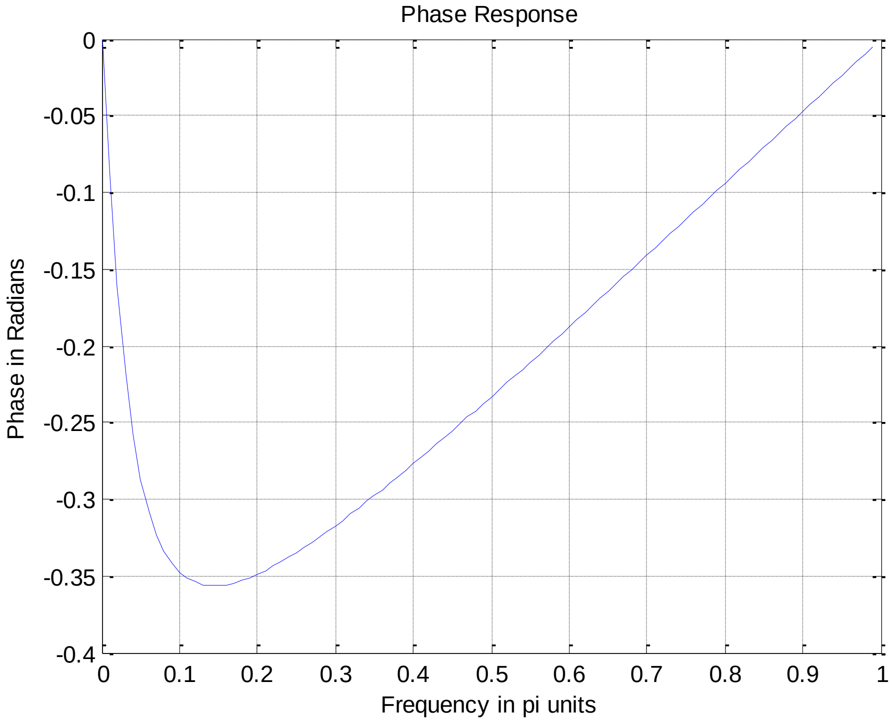
$$\left|H\left(e^{jw}\right)\right| = \sqrt{\frac{1}{\left(1-0.9\cos(w)\right)^2+\left(0.9\sin(w)\right)^2}} = \sqrt{\frac{1}{1-2\times(0.9)\times\cos(w)+(0.9)^2\times\left[\left(\cos(w)\right)^2+\left(\sin(w)\right)^2\right]}}$$

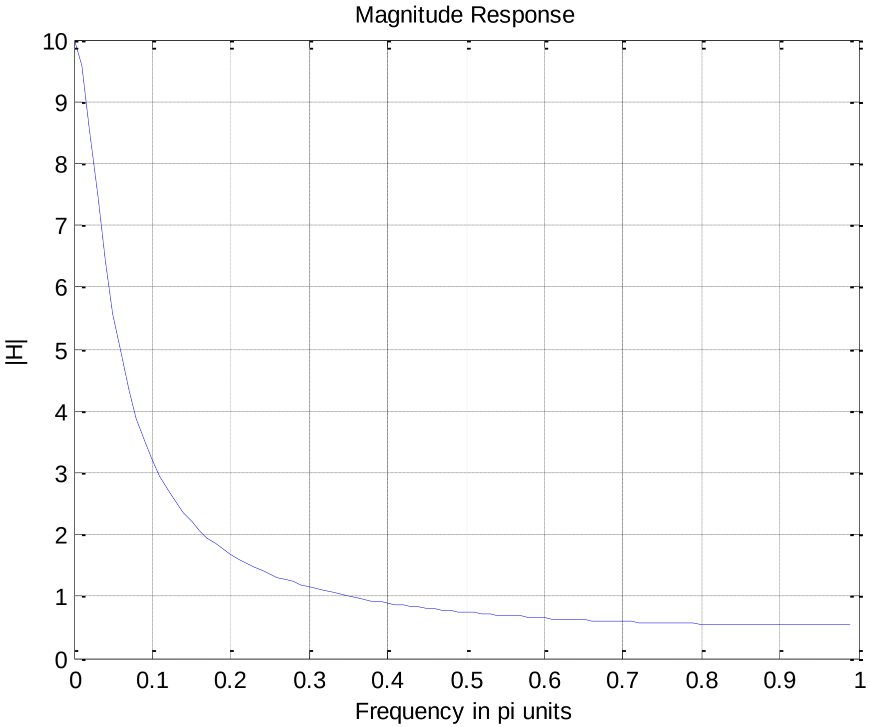
$$\left|H\left(e^{jw}\right)\right| = \sqrt{\frac{1}{1.81-1.8\times\cos(w)}} =$$

and

$$\angle H\left(e^{jw}\right) = -\arctan\left[\frac{0.9\sin(w)}{1-0.9\cos(w)}\right]$$

b- The magnitude and phase plots





EXAMPLE 3.3

A digital system is specified by the difference equation as:

$$y(n) = (0.8)y(n-1) + x(n)$$

a- Determine $|H(e^{jw})|$

Solution :

a- Take the DTFT for both sides or apply the above equation:

$$Y(e^{jw}) = (0.8)Y(e^{jw})e^{-jw} + X(e^{jw})$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1-(0.8)e^{-jw}}$$

EXAMPLE 3.4 (H.W.)

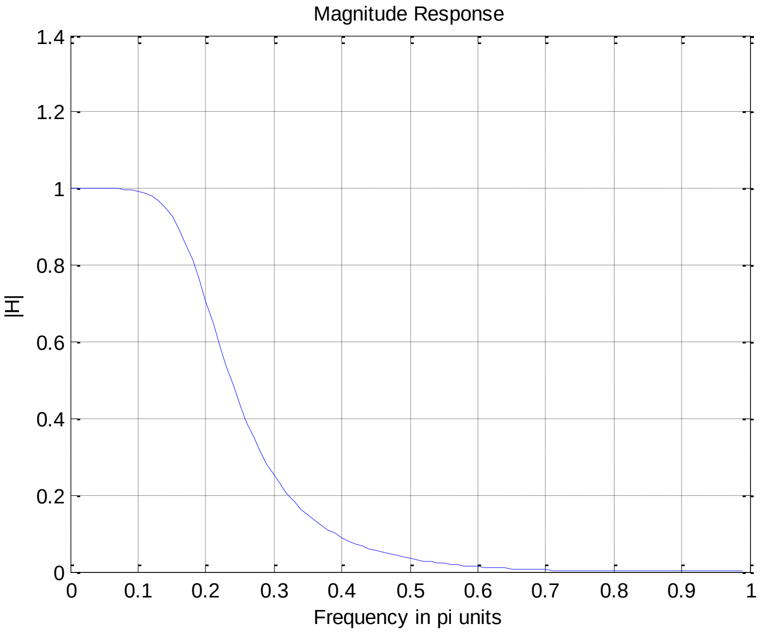
A third order low pass filter is described by the difference equation:

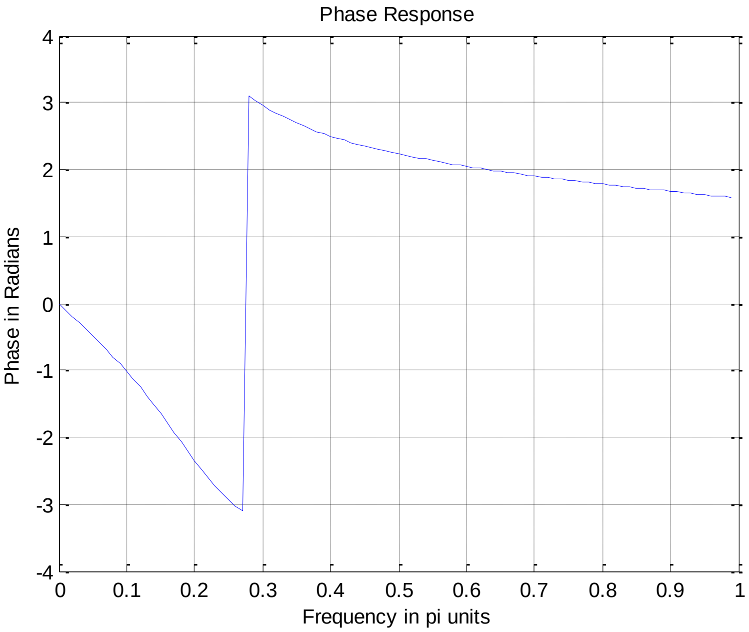
$$y(n)=0.081x(n)+0.0543x(n-1)+0.0543x(n-2)+0.0181x(n-3)$$
$$+1.76y(n-1)-1.1829y(n-2)+0.27814y(n-3)$$

a- Determine $H(e^{jw})$

b- Plot the magnitude and phase response of the filter.

Solution





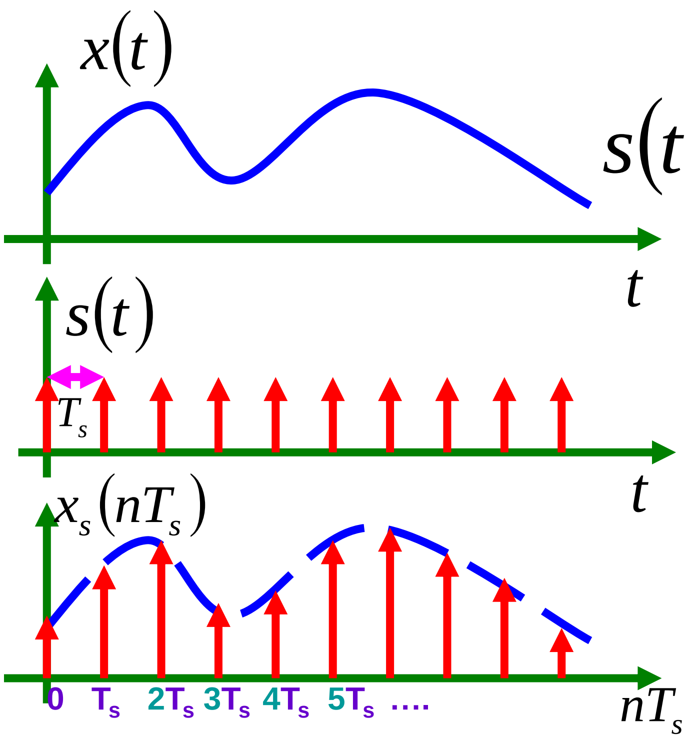
3.4 SAMPLING OF ANALOG SIGNALS

- **The sequence *x(nT)*** is obtained from **a continuous time signal x(t)** by **sampling**
- **This is done multiplying a periodic impulse train s(t) [sampling function] by x(t)**
- **The period T_s is called the sampling period and $F_s = 1/T_s$ is the sampling frequency**

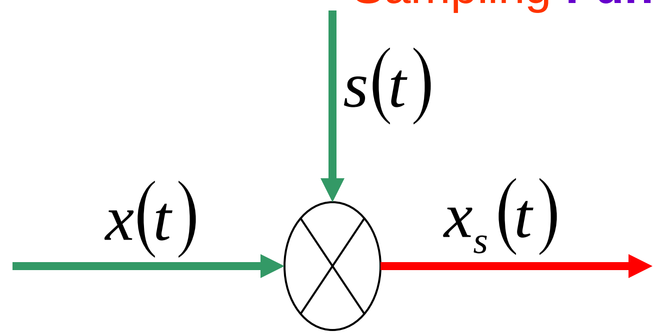
$$w_s = 2\pi F_s$$

$$x_s(t) = s(t).x(t)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



Sampling **Function**



Therefore, the sampled signal is given by :

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

From the multiplicative property, $X_s(w)$ **is the convolution of**

$$X(jw) \text{ and } \delta(jw)$$

Therefore,

$$X_s(jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)S(j(w-\theta))d\theta$$

It is known that

$$S(jw) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(w - kw_s)$$

Since, the convolution with **an impulse** simply **shifts a signal**

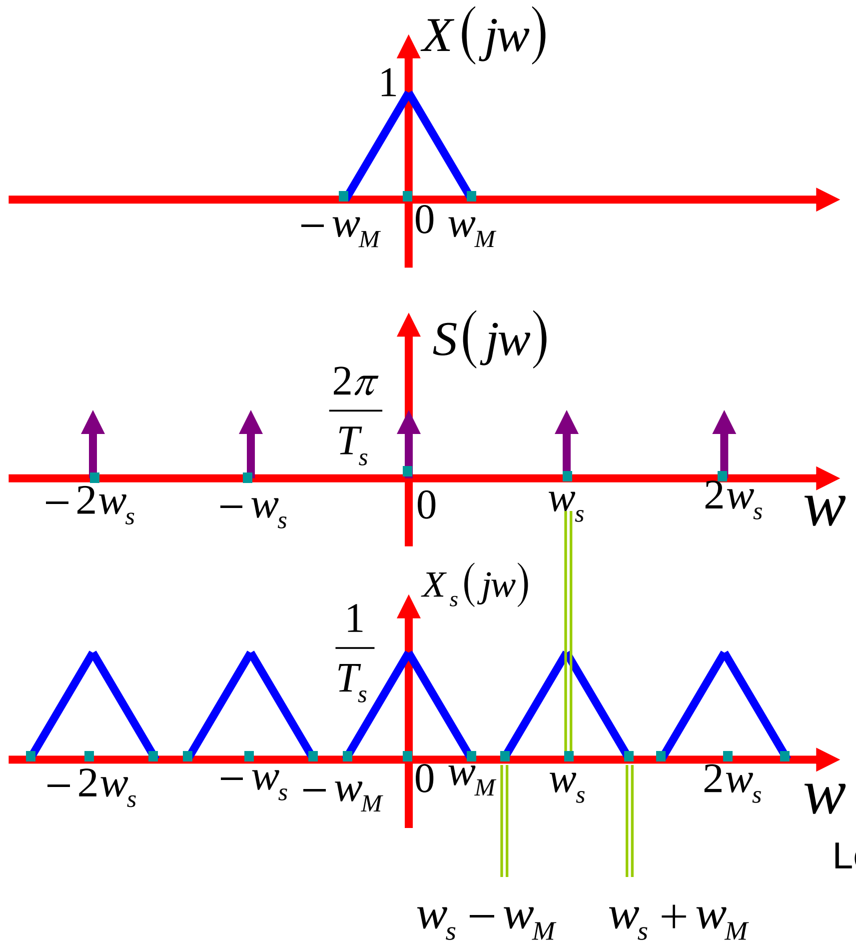
$$X(jw) * \delta(w - w_o) = X(j(w - w_o)) \Rightarrow X_s(jw) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(w - kw_s))$$

$$X_s(jw) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(w - kw_s)\big)$$

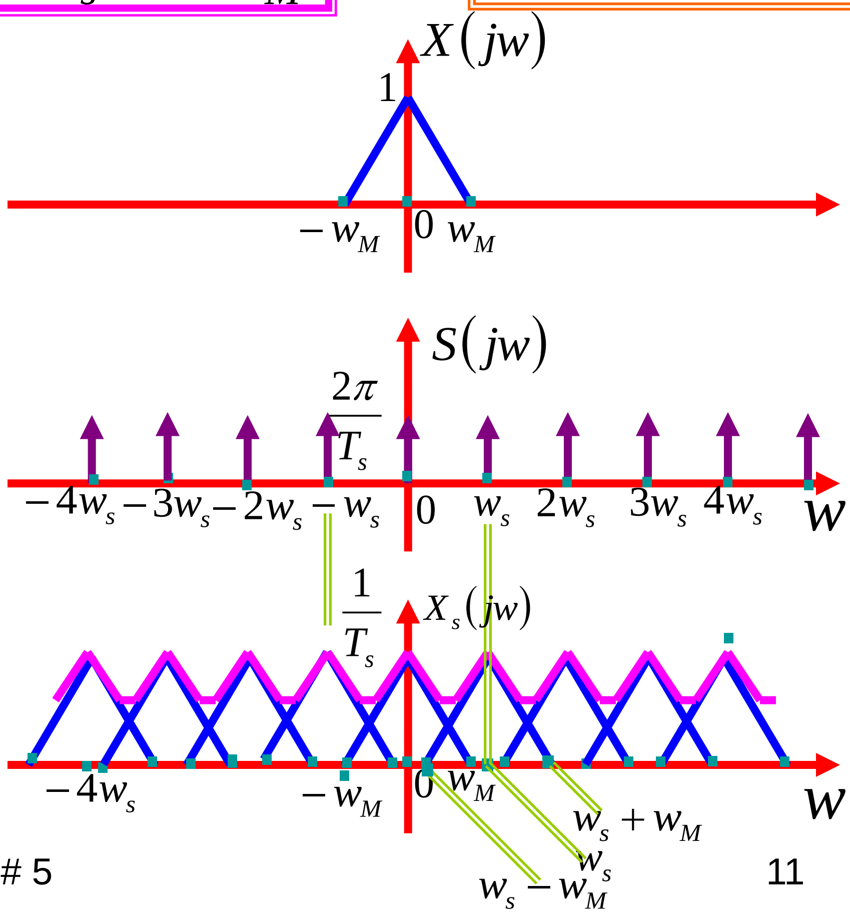
That is $X_s(jw)$ Is a periodic function of w consisting of :

superposition of shifted replicas of $X_s(jw)$ scaled by $\frac{1}{T_s}$

$w_s \geq 2w_M \Rightarrow w_s - w_M \geq w_M$



$w_s \prec 2w_M \Rightarrow w_s - w_M \prec w_M$

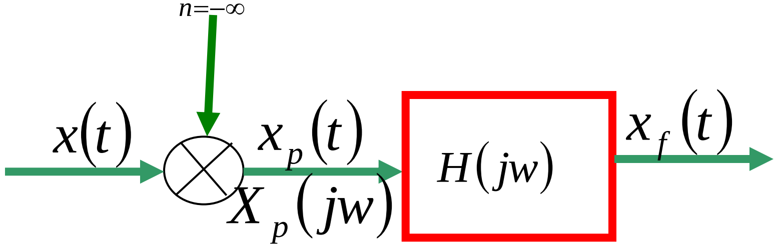


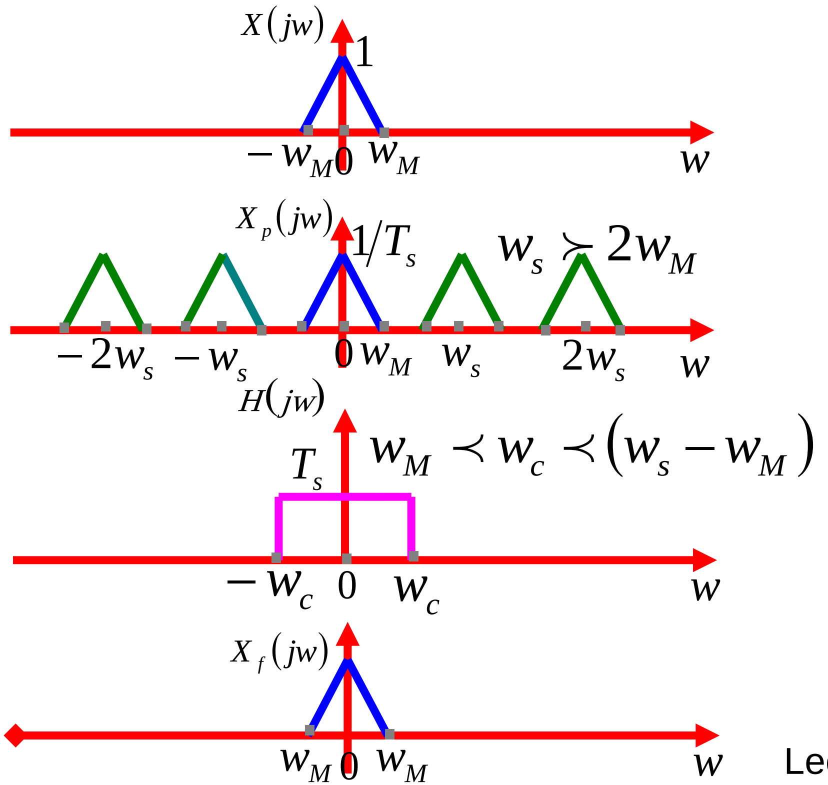
If $w_s \succ 2w_M$ **The signal can be recovered exactly from** $x_s(t)$

by means of → **Low pass filter with gain** T_s

→ **Cut off frequency greater than** w_M

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$





$$\Downarrow$$

$$H(e^{jw})=\begin{cases} T_s & |w|\le \frac{w_s}{2} \\ 0 & otherwise \end{cases}$$

3.5 SAMPLING THEOREM

Let $x(t)$ be a band –limited signal with $X(jw) = 0 \text{ for } |w| \succ w_M$.
Then $x(t)$ Is uniquely determined by its samples $x(nT_s), n = 0, \pm 1, \pm 2, \ldots\ldots \text{if}$

Where, $w_s \succ 2w_M$ and $w_s = \frac{2\pi}{T_s}$

- **Given these samples**, we can **reconstruct x(t)** by generating a **periodic impulse train.**
- **The successive impulses** have amplitudes that are **successive sample values**.
- **This impulse train** is then **processed through an ideal low pass filter** with gain T_s and **cut off frequency greater than** w_s and **less than** $w_s - w_M$.
- **The resulting output signal** will **exactly equal x(t).**