Electric Engineering II
EE 326
Lecture 9
Stability

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Understanding Poles and Zeros

The transfer function provides a basis for determining important system response characteristics without solving the complete differential equation

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$H(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2) \ldots (s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_{n-1})(s - p_n)},$$

$$H(s) = \frac{N(s)}{D(s)} = K\frac{(s-z_1)(s-z_2)\ldots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_{n-1})(s-p_n)},$$

$K = b_m/a_n$

the z_i's are the roots of the equation

$$N(s) = 0,$$

and are defined to be the system *zeros*,
and the p_i's are the roots of the equation

$$D(s) = 0,$$

and are defined to be the system *poles*.

Example

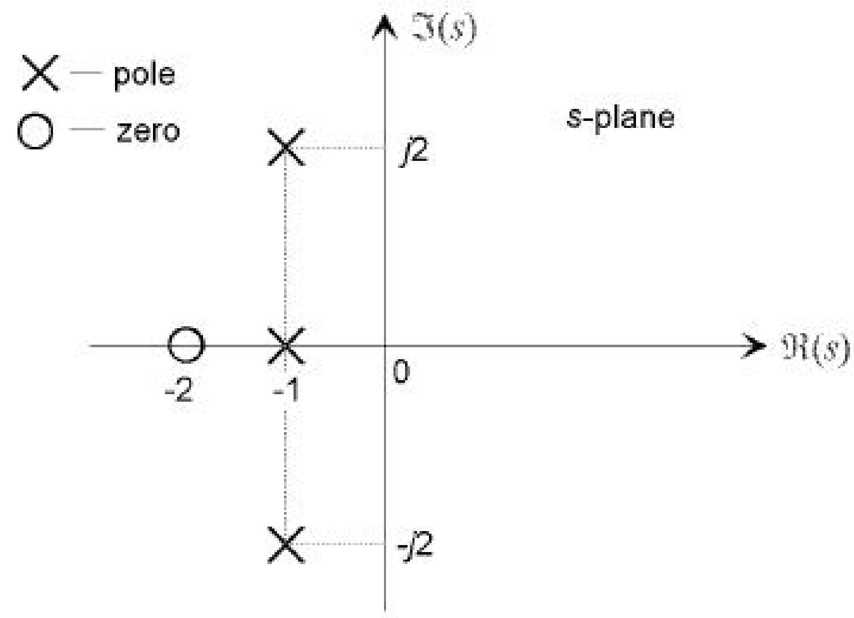
A system has a pair of complex conjugate poles $p1$, $p2 = -1 \pm j2$, a single real zero $z1 = -4$, and a gain factor $K = 3$. Find the differential equation representing the system.

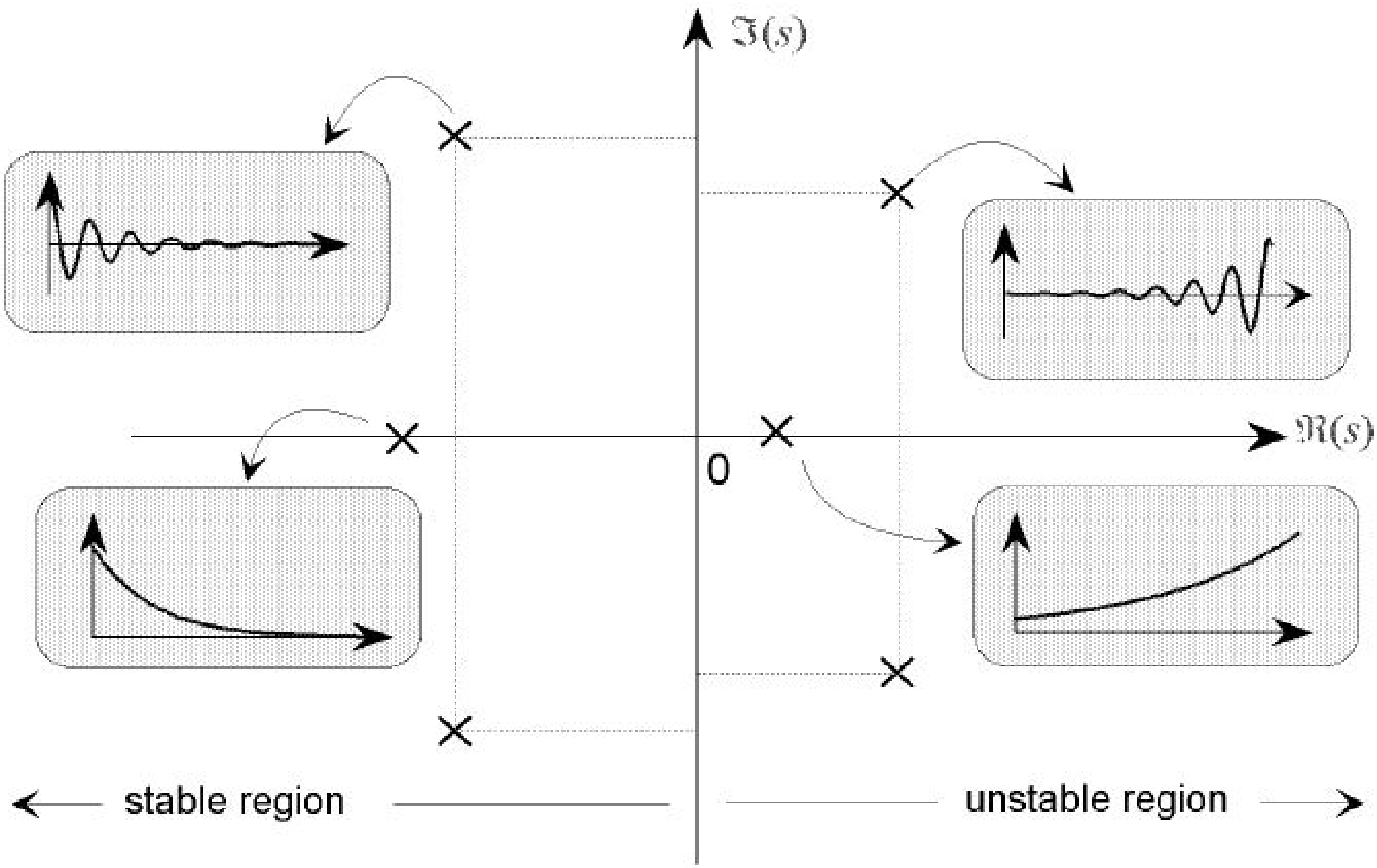
$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\frac{du}{dt} + 12u$$

the poles and zeros of a transfer function may be complex, and the system dynamics may be represented graphically by plotting their locations on the complex *s*-plane, whose axes represent the real and imaginary parts of the complex variable *s*. Such plots are known as *pole-zero plots*.

$$H(s) = \frac{(3s+6)}{(s^3+3s^2+7s+5)}$$

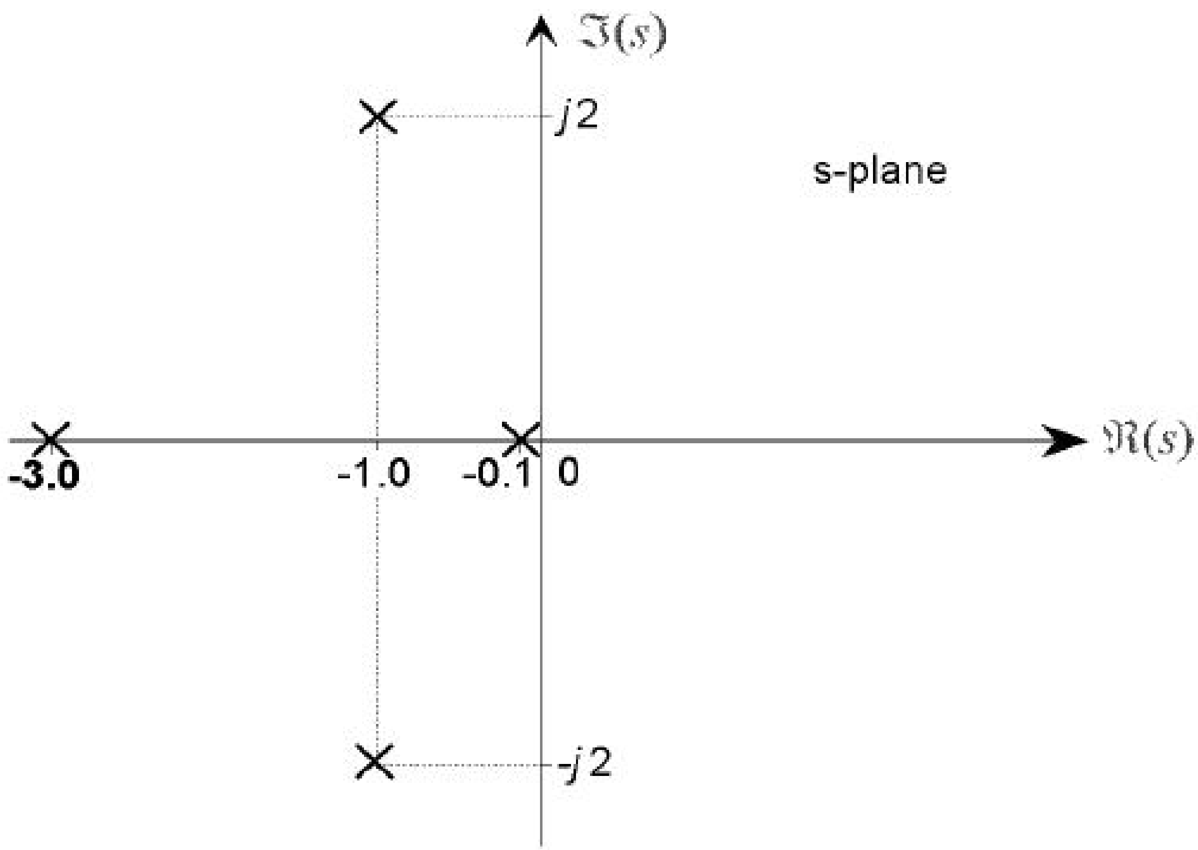
$$= 3\frac{(s-(-2))}{(s-(-1))(s-(-1-2j))(s-(-1+2j))}$$




ℑ(s)
ℜ(s)
0
stable region
unstable region

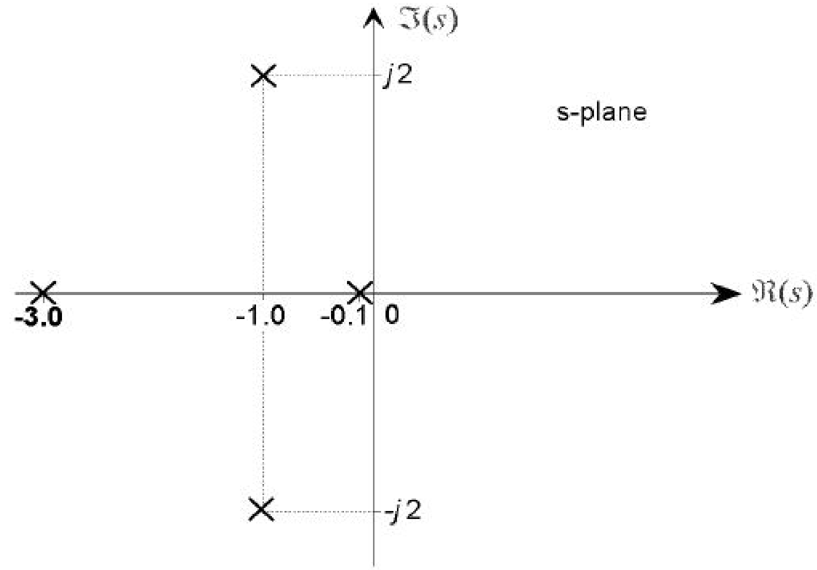
Example

Comment on the expected form of the response of a system with a pole-zero plot shown in Fig.



The two real poles correspond to decaying exponential terms

$$C_1 e^{-3t} \text{ and } C_2 e^{-0.1t}$$



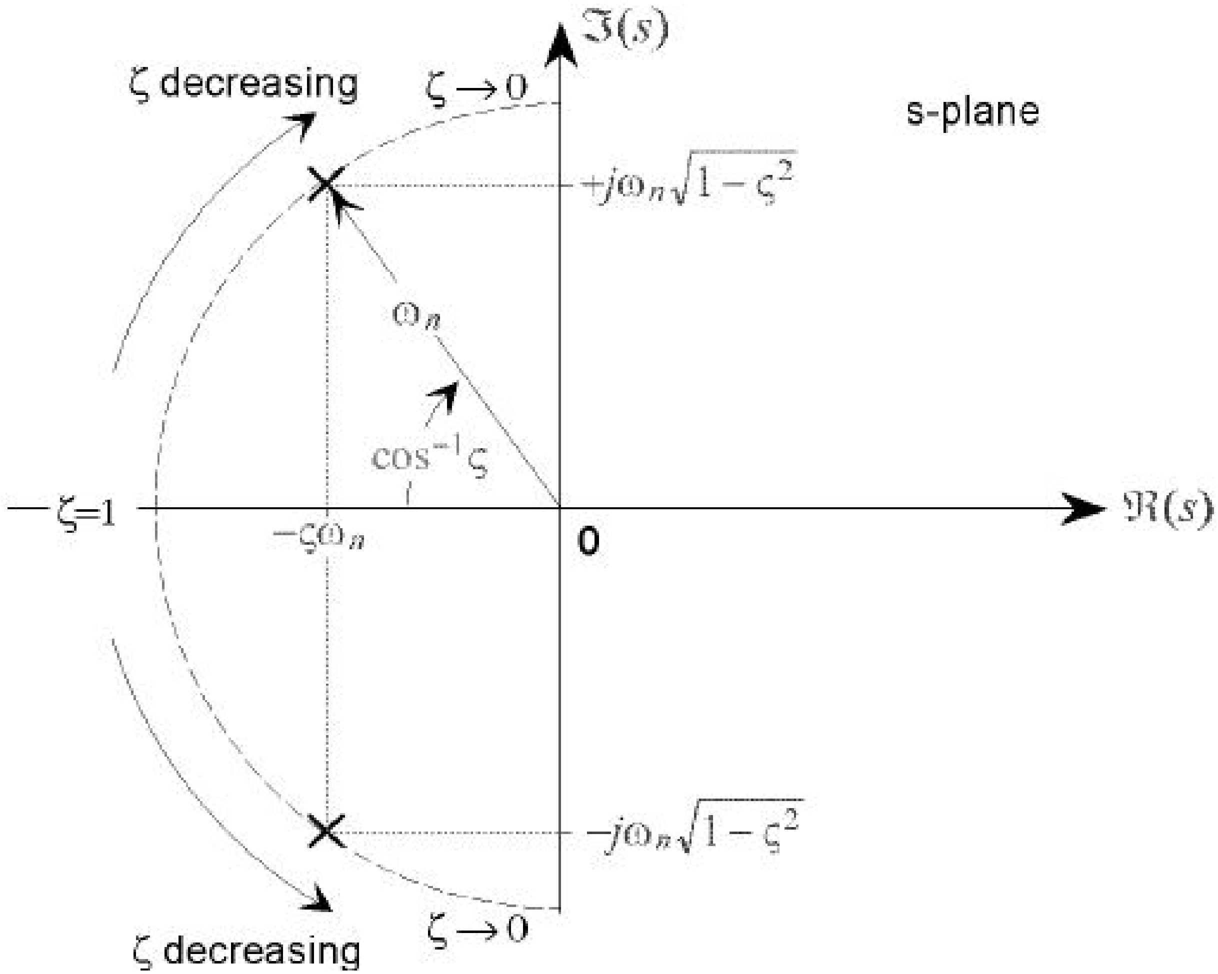
and the complex conjugate pole pair introduce an oscillatory component

$$Ae^{-t}\sin(2t+\phi)$$

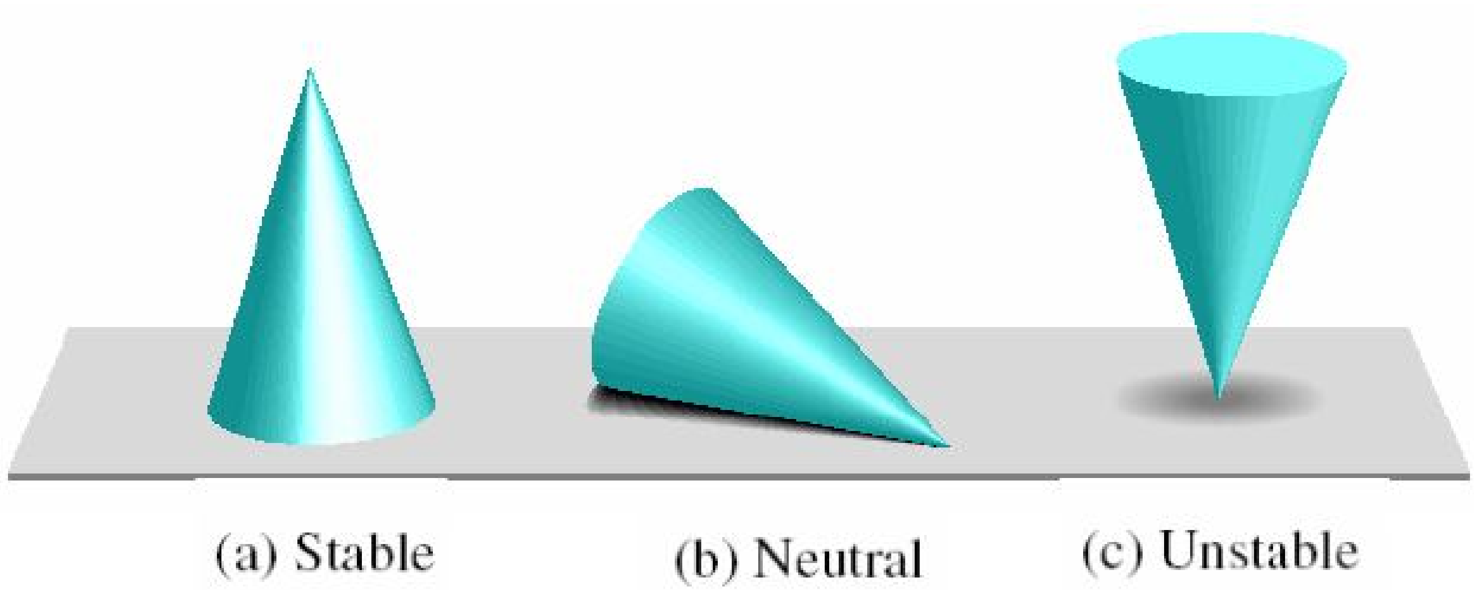
so that the total homogeneous response is

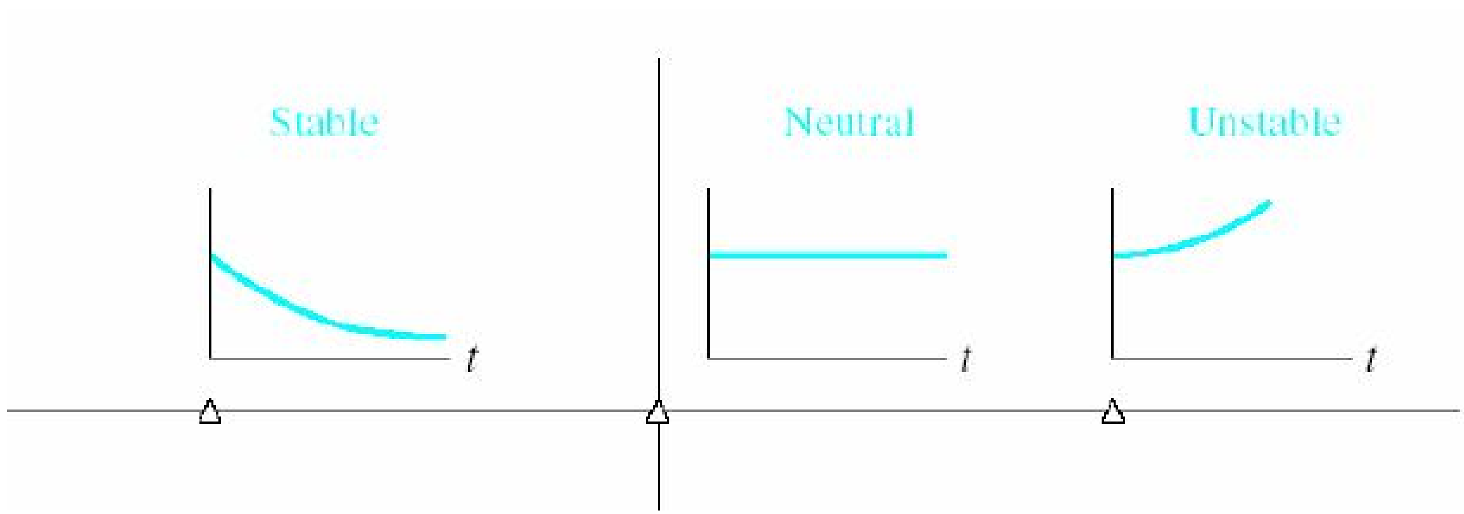
$$y_h(t) = C_1 e^{-3t} + C_2 e^{-0.1t} + Ae^{-t}\sin(2t+\phi)$$

1. The term e^{-3t}, with a time-constant τ of 0.33 seconds, decays rapidly and is significant only for approximately 4τ or 1.33seconds.
2. The response has an oscillatory component $Ae^{-t}\sin(2t + \phi)$ defined by the complex conjugate pair, and exhibits some overshoot. The oscillation will decay in approximately four seconds because of the e^{-t} damping term.
3. The term $e^{-0.1t}$, with a time-constant $\tau = 10$ seconds, persists for approximately 40 seconds. It is therefore the *dominant* long term response component in the overall homogeneous response.

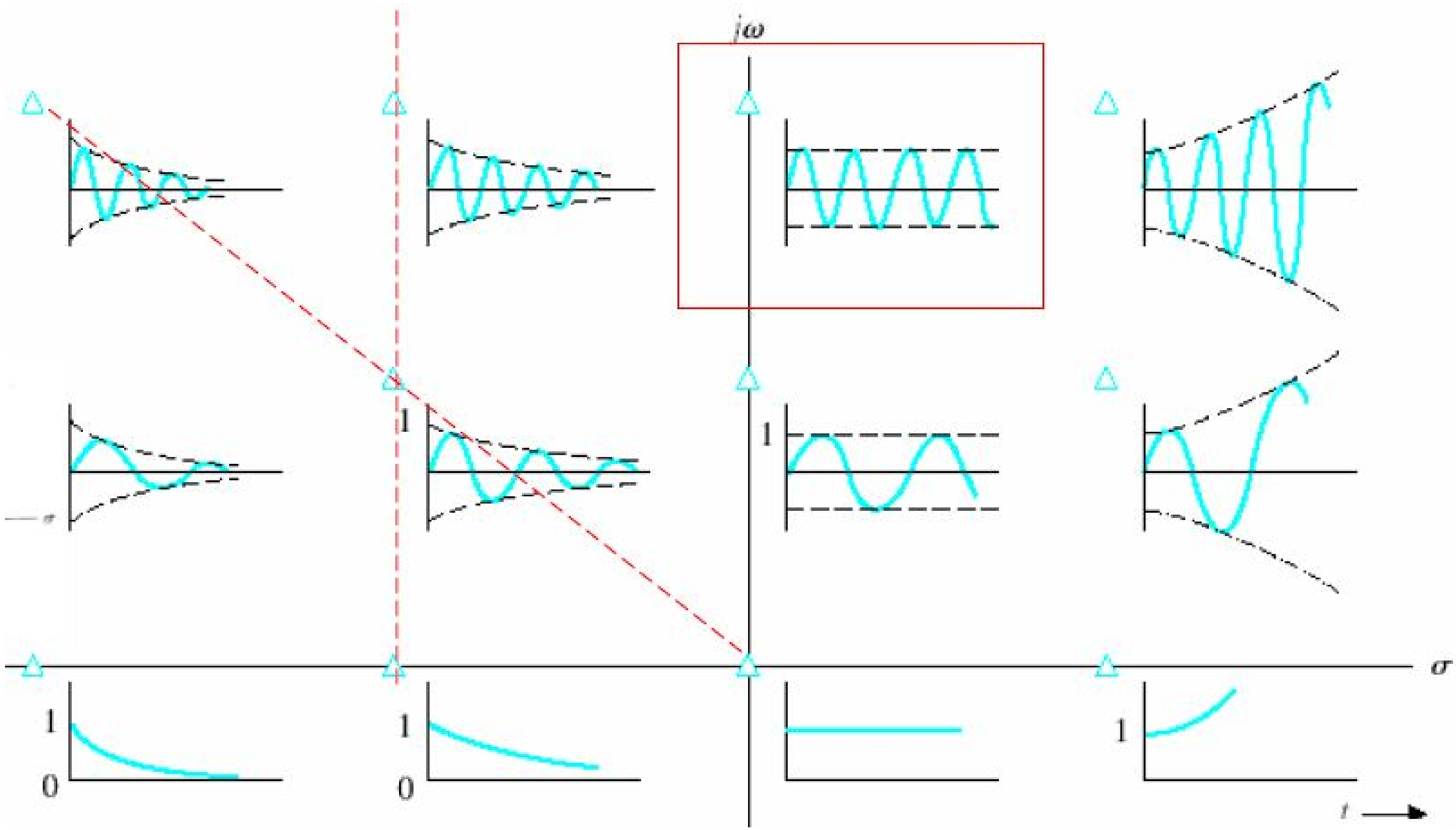

ζ decreasing
ζ → 0
ℑ(s)
s-plane
+jωn√1 − ς²
ωn
cos⁻¹ς
ζ=1
−ςωn
0
ℜ(s)
−jωn√1 − ς²
ζ decreasing
ζ → 0

The Concept of Stability





S-Plane and Transient Response



Stable

Unstable

Definitions of Stability

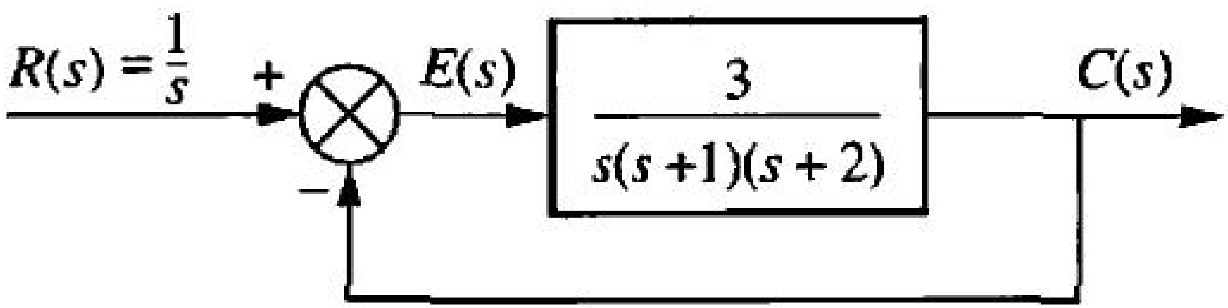
- **BIBO stability:** A system is said to be BIBO stable if for any bounded input, its output is also bounded.

- **Absolute stability:** Stable /Unstable

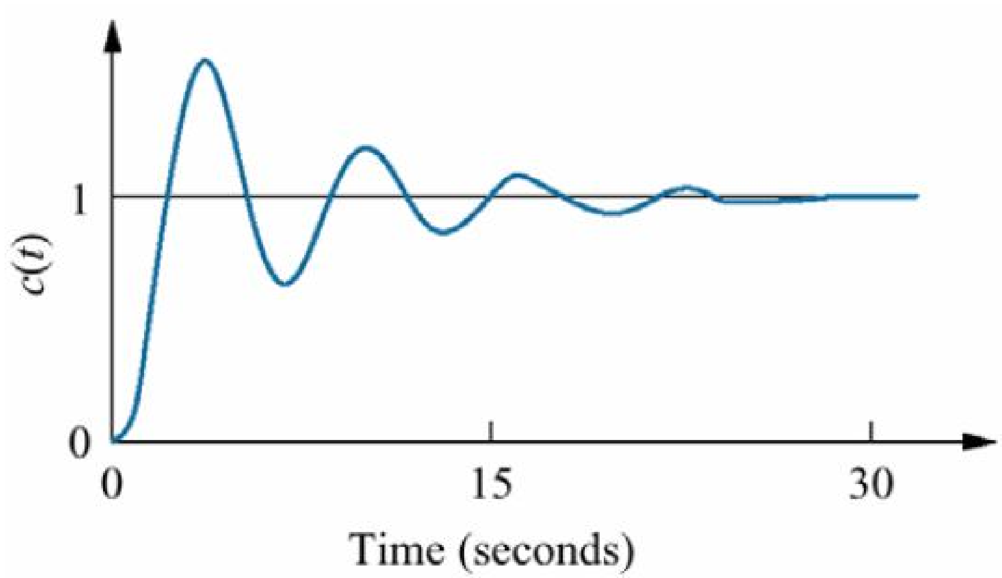
- **Relative stability:** Degree of stability (i.e.how far from instability)

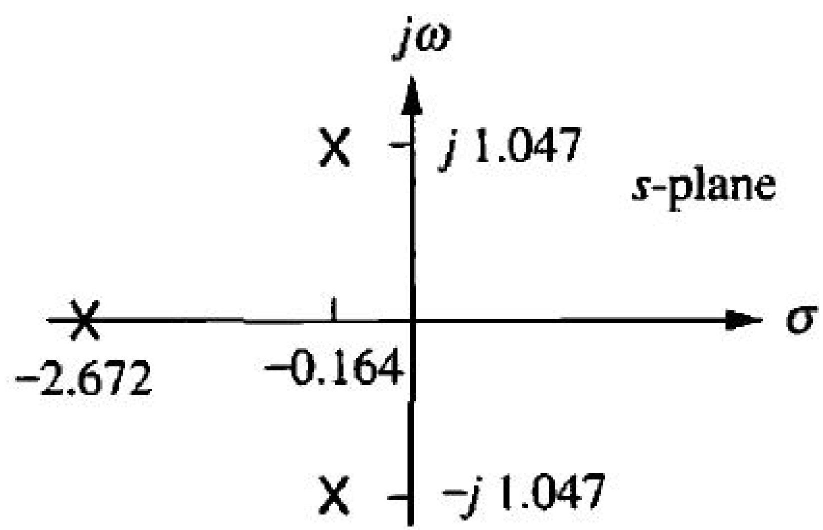
- A **stable** linear system described by a T.F.is such that **all** its **poles** have **negative real parts**

Feedback and Stability

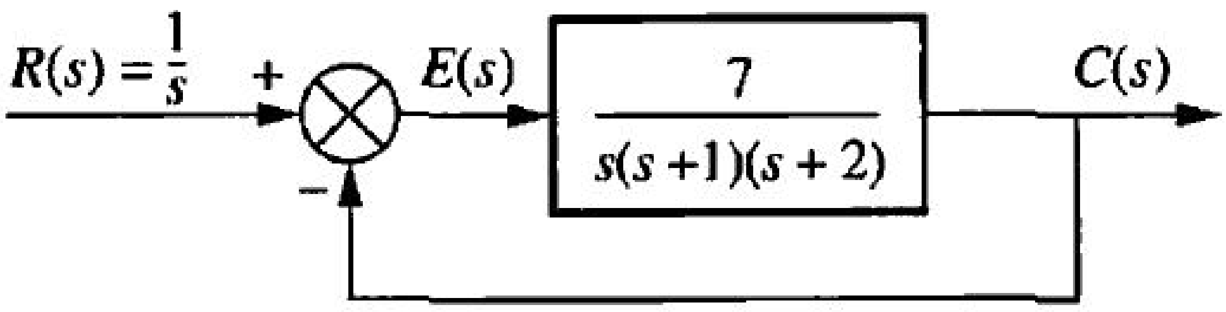


Stable system

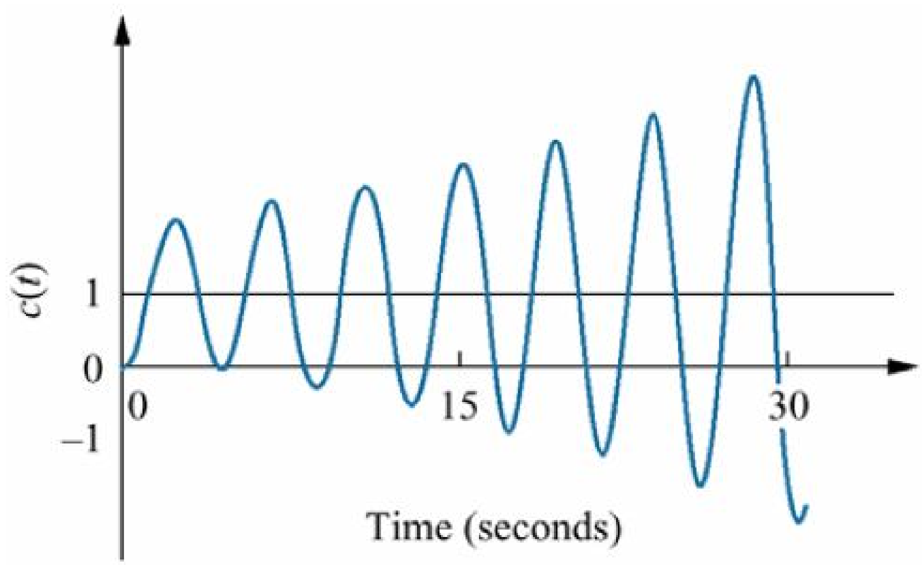


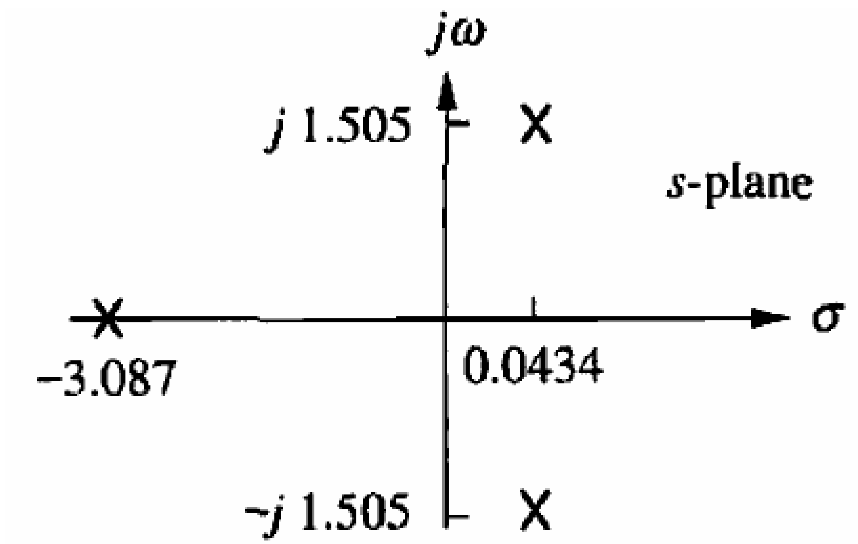


Feedback and Stability



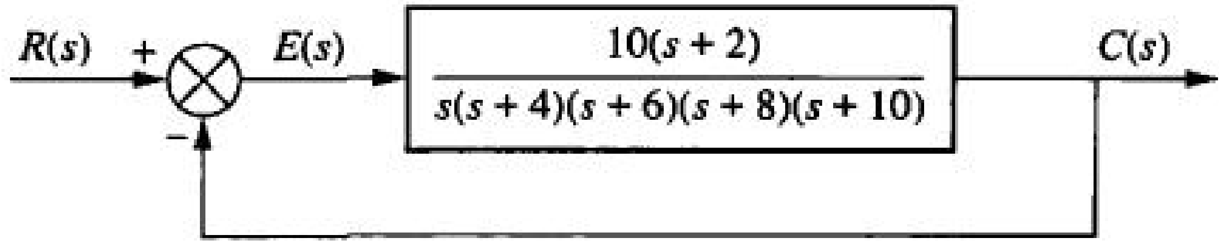
Unstable system



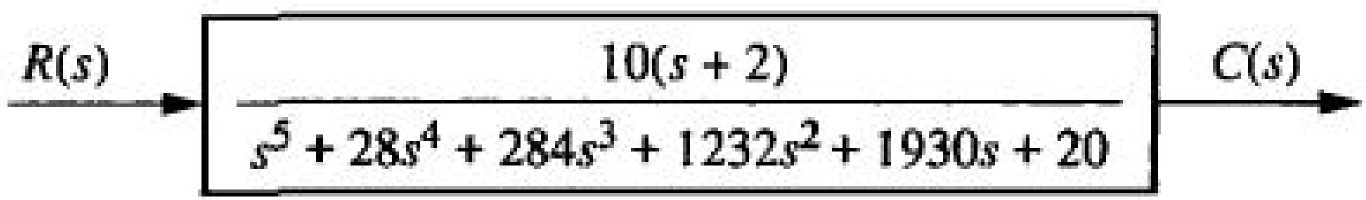


For a stable system, the roots of the characteristic equation must have negative real parts.

It is not always a simple matter to determine if a feedback control system is stable. Unfortunately, a typical problem that arises is shown



Although we know the poles of the forward transfer function, we do not know the location of the poles of the equivalent closed-loop system without factoring or otherwise solving for the roots.



Routh-Hurwitz Criterion

Edward Routh, 1831 (Quebec)-1907 (Cambridge, England)

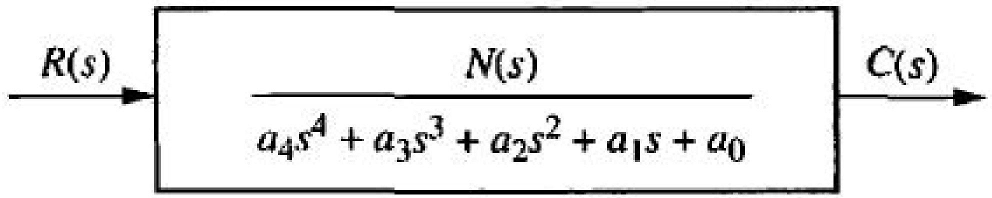
Adolf Hurwitz, 1859 (Germany)-1919 (Zurich)

Routh-Hurwitz Criterion

A method that yields stability information without the need to solve for the closed-loop system poles. Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the ;'w-axis. ***(Notice that we say how many, not where.)*** We can find the number of poles in each section of the s-plane, but we cannot find their coordinates. The method is called the ***Routh-Hurwitz criterion*** for stability

The method requires two steps:
(1) Generate a data table called a *Routh table* and
(2) interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on *the jw-axis.*

Generating a Basic Routh Table



Since we are interested in the system poles, we focus our attention on the denominator. We first create the Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Begin by labeling the rows with powers of *s* from the highest power of the denominator of the closed-loop transfer function

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

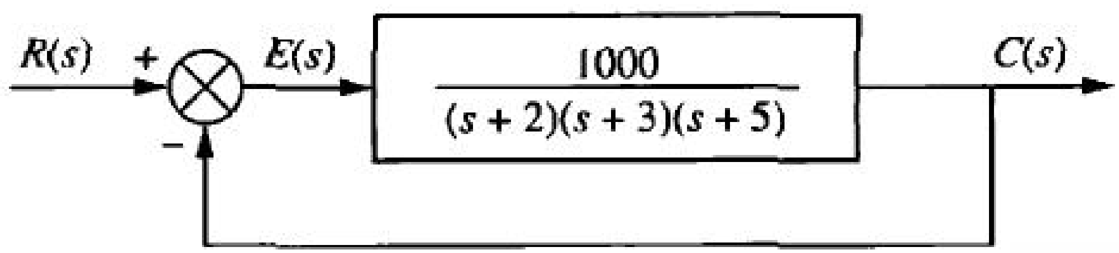
Next start with the coefficient of the highest power of *s* in the denominator and list, horizontally in the first row, every other coefficient.

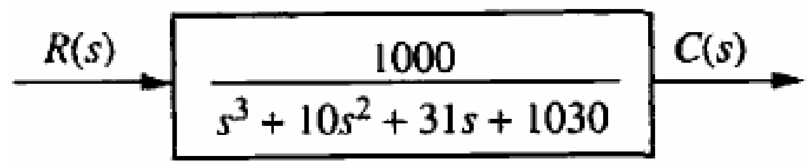
In the second row, list horizontally, starting with the next highest power of *s,* every coefficient that was skipped in the first row.

TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Example 1





s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix}1 & 31\\ 1 & 103\end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix}1 & 103\\ -72 & 0\end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$

If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.

Example 2

$$P(s) = 3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$$

Routh-Hurwitz Criterion: Special Cases

Zero Only in the First Column

If the first element of a row is zero, division by zero would be required to form the next row. To avoid this phenomenon, an epsilon, €, is assigned to replace the zero in the first column. The value *e* is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined.

Stability via Epsilon Method

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

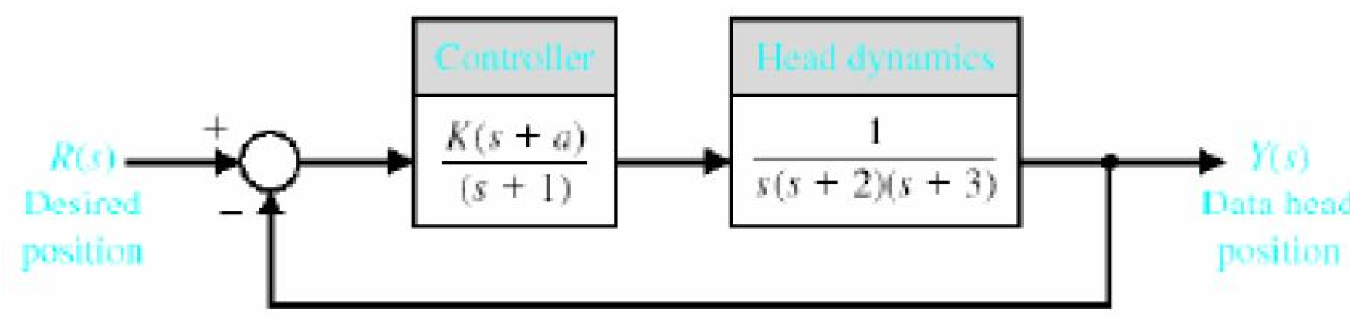
				$\epsilon = +$	$\epsilon = -$
s^5	1	3	5	+	+
s^4	2	6	3	+	+
s^3	~~0~~ ϵ	$\frac{7}{2}$	0	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0	+	+
s^0	3	0	0	+	+

$$D(s) = s^3 + 3s^2 + 3s + (1+k)$$

s^3	1	3
s^2	3	$1+k$
s^1	$b_1 = (8-k)/3$	
s^0	$c_1 = 1+k$	

For Stable system

$$\left.\begin{aligned} \frac{8-k}{3} > 0 \Rightarrow 8 > k \\ 1+k > 0 \Rightarrow k > -1 \end{aligned}\right\} \Rightarrow 8 > k > -1$$



$$1 + GC(s) = 1 + \frac{K(s+a)}{s(s+1)(s+2(s+3)} = 0$$

$$\begin{array}{c|ccc} s^4 & 1 & 11 & Ka \\ s^3 & 6 & (K+6) & \end{array}$$

where $b_3 = \dfrac{60-K}{6}$ and $c_3 = \dfrac{b_3(K+6) - 6Ka}{b_3}$

$b3 > 0 \quad \Rightarrow \quad \boxed{K < 60}$

$(60-K)(K+6) - 36Ka > 0 \quad \Rightarrow \quad \boxed{a < \dfrac{(60-K)(K+6)}{36K}}$

$\boxed{\text{With } K = 40,\ a < 0.639}$