Modern Control
EE 419
Lecture 8

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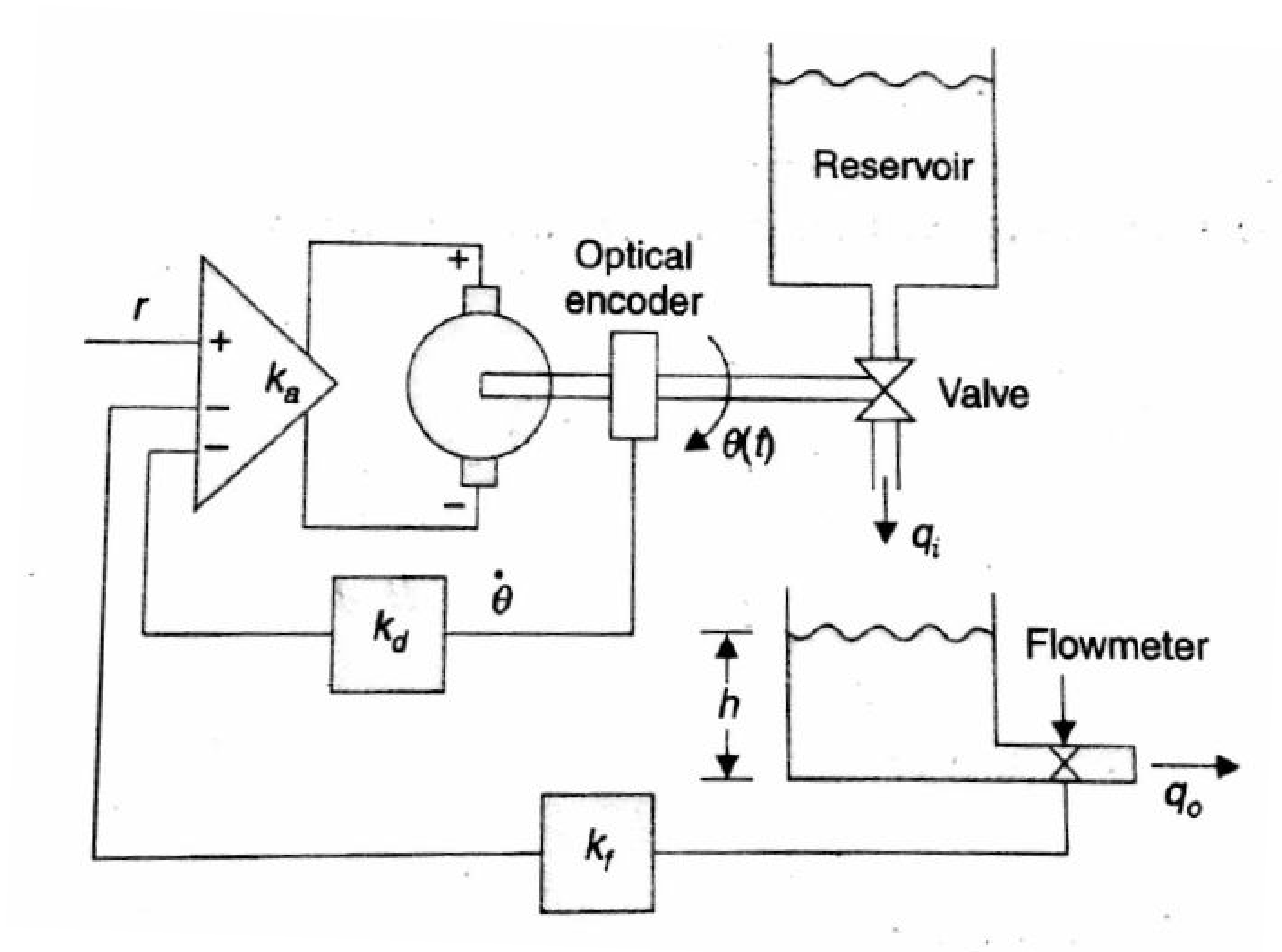
Control Flow

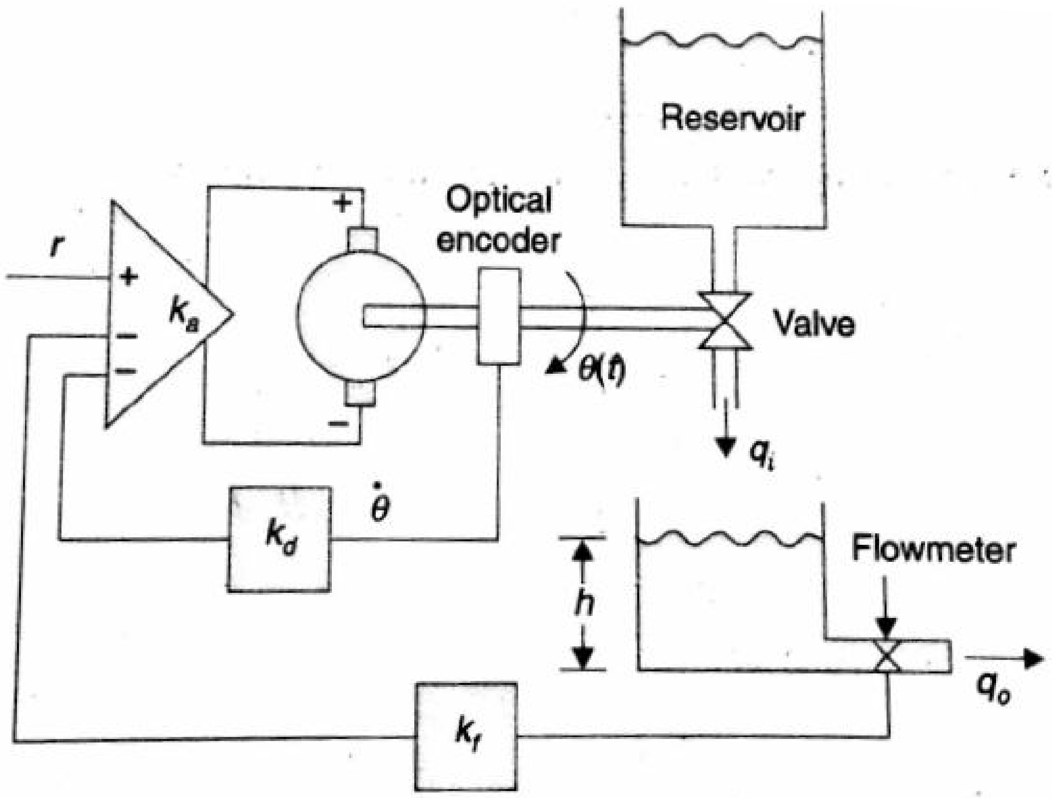


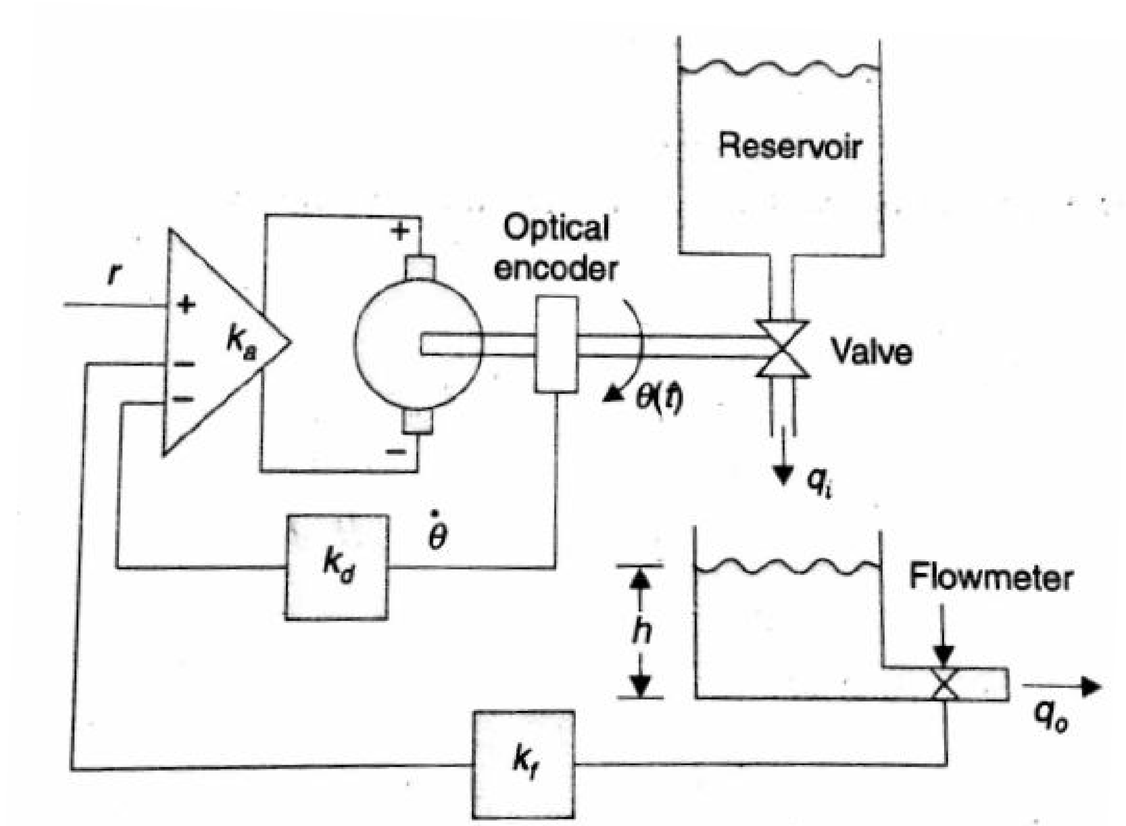
$k_a = 25,\ k_p = 1,\ k_d = 0.005$

$k_m = 5,\ J = 0.05,\ R_a = 1\ \Omega$

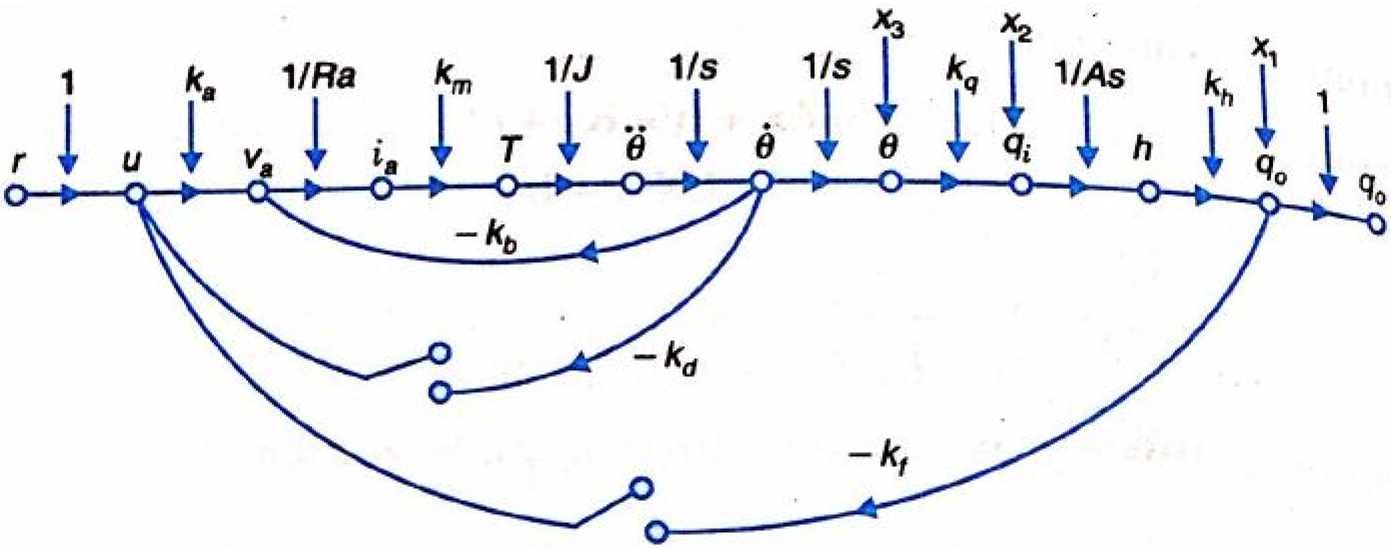
$q_i = k_q\theta,\ k_q = 8$, tank area $A = 50\ \text{m}^2$

$q_0 = k_h h,\ k_h = 225,\ k_f = 0.25.$





$$x_1 = q_0,\ x_2 = q_i,\ x_3 = \theta$$



$$\dot{x}_i = \frac{k_h}{A} x_2$$

$$\dot{x}_2 = k_q x_3$$

$$\dot{x}_3 = (rk_a - k_b x_3)\frac{k_m}{R_a J}$$

$$= \left(\frac{k_a k_m}{R_a J}\right) r - \left(\frac{k_b k_m}{R_a J}\right) x_3$$

$$y = q_0 = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_h}{A} & 0 \\ 0 & 0 & k_q \\ 0 & 0 & -\left(\frac{k_b k_m}{R_a J}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{vmatrix} 0 \\ 0 \\ \frac{k_a k_m}{R_a J} \end{vmatrix} [u] \; ; \; u = r$$

$$y = [1\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Controllability and Observability

Determine this state from the observation of the output over a finite time interval. The concepts of ***controllability and observability*** were introduced by Kalman. They play an important role in the design of control systems in state space. In fact, the conditions of ***controllability and observability*** may govern the existence of a complete solution to the control system design problem.

CONTROLLABILITY

A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $\mathbf{x}(t_0)$ to any other state in a finite interval of time.

Complete State Controllability of Continuous-Time Systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where $\mathbf{x}$ = state vector (n-vector)

u = control signal (scalar)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times 1$ matrix

The system described by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ is said to be state controllable at t=t0 if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $0 \leq t \leq t_1$
If every state is controllable, then the system is said to be completely state controllable.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau$$

Applying the definition of complete state controllability just given,

$$\mathbf{x}(t_1) = \mathbf{0} = e^{\mathbf{A}t_1}\mathbf{x}(0) + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}u(\tau)\,d\tau$$

$$\mathbf{x}(0) = -\int_0^{t_1} e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)\,d\tau \quad \Longrightarrow \quad \mathbf{x}(0) = -\sum_{k=0}^{n-1}\mathbf{A}^k\mathbf{B}\beta_k$$

$$\mathbf{x}(0) = -\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{B} \beta_k$$

$$= -\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right] \begin{bmatrix} \beta_0 \\ \hdashline \beta_1 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline \beta_{n-1} \end{bmatrix}$$

The system is completely state controllable if and only if the vectors $\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}$ are linearly independent

$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$ is of rank n

Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Since

$$[\mathbf{B} \,\vdots\, \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{singular}$$

the system is not completely state controllable.

Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$[\mathbf{B} \,\vdots\, \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \text{nonsingular}$$

Complete Observability of Continuous-Time Systems.

A system is said to be observable at time t0 if, with the system in state x(t0), it is possible to determine this state from the observation of the output over a finite time interval.

$$\mathbf{y} = \mathbf{C}\mathbf{x} \quad \Longrightarrow \quad \mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{A}^k \quad \Longrightarrow \quad \mathbf{y}(t) = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{C}\mathbf{A}^k\mathbf{x}(0)$$

$$\mathbf{y}(t) = \alpha_0(t)\mathbf{C}\mathbf{x}(0) + \alpha_1(t)\mathbf{C}\mathbf{A}\mathbf{x}(0) + \cdots + \alpha_{n-1}(t)\mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(0)$$

If the system is completely observable, then, given the output y(t) over a time interval $0 \leq t \leq t_1$ x(0) is uniquely determined from Equation

It can be shown that this requires the rank of the nm*r matrix

$$\begin{bmatrix} \mathbf{C} \\ \hdashline \mathbf{CA} \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline \mathbf{CA}^{n-1} \end{bmatrix}$$

to be n.

The system is completely observable if and only if the n*nm matrix

$$\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

is of rank n or has n linearly independent column vectors. This matrix is called the observability matrix.

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left[\mathbf{B} \ \vdots \ \mathbf{AB}\right] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\left[\mathbf{CB} \ \vdots \ \mathbf{CAB}\right] = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\left[\mathbf{C}^* \ \vdots \ \mathbf{A}^*\mathbf{C}^*\right] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Show that the following system is not completely observable:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [4 \quad 5 \quad 1]$$

$$[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; (\mathbf{A}^*)^2\mathbf{C}^*] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

In fact, in this system, cancellation occurs in the transfer function of the system. The transfer function between X1(s) and U(s) is

$$\frac{X_1(s)}{U(s)} = \frac{1}{(s+1)(s+2)(s+3)}$$

and the transfer function between Y(s) and X1(s) is

$$\frac{Y(s)}{X_1(s)} = (s+1)(s+4)$$

$$\frac{Y(s)}{U(s)} = \frac{(s+1)(s+4)}{(s+1)(s+2)(s+3)}$$

Alternative Form of the Condition for Complete State Controllability.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times r$ matrix

If the eigenvectors of **A** are distinct, then it is possible to find a transformation matrix **P** such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \lambda_n \end{bmatrix}$$

Diagonal Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

$\mathbf{x} = \mathbf{P}\mathbf{z}$ Substituting in $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}$$

By defining

$$\mathbf{P}^{-1}\mathbf{B} = \mathbf{F} = (f_{ij})$$

$$\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + f_{11}u_1 + f_{12}u_2 + \cdots + f_{1r}u_r \\
\dot{z}_2 &= \lambda_2 z_2 + f_{21}u_1 + f_{22}u_2 + \cdots + f_{2r}u_r \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{z}_n &= \lambda_n z_n + f_{n1}u_1 + f_{n2}u_2 + \cdots + f_{nr}u_r
\end{aligned}$$

If the elements of any one row of the n*r matrix F are all zero, then the corresponding state variable cannot be controlled by any of the ui