Automatic Control
EE 418
Lecture 2

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Differential equation of physical systems

Linear Ordinary Differential Equations

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

which is also known as a **linear ordinary differential equation** if the coefficients $a_0, a_1, \ldots, a_{n-1}$ are not functions of $y(t)$.

A first-order linear ordinary differential equation

$$\frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

Second-order differential equation

Nonlinear Differential Equations

Many physical systems are nonlinear and must be described by nonlinear differential equations. For instance, the following differential equation that describes the motion of a pendulum of mass *m* and length /, later discussed in this chapter, is

$$m\ell\frac{d^2\theta(t)}{dt^2} + mg\sin\theta(t) = 0$$

Because *6{t)* appears as a sine function, Eq. (2-100) is nonlinear, and the system is called a **nonlinear system.**

Modeling of Dynamic Systems

One of the most important tasks in the analysis and design of control systems is mathematical modeling of the systems. One of the most common methods of modeling linear systems is the transfer function method.

The control systems engineer often has the task of determining not only how to accurately describe a system mathematically but, more importantly, how to make proper assumptions and approximations, whenever necessary, so that the system may be realistically characterized by a linear mathematical model.

A control system may be composed of various components including mechanical, thermal, fluid, pneumatic, and electrical; sensors and actuators; and computers. In this chapter, we review basic properties of these systems, otherwise known as **dynamic systems.**

The main objectives of this Lecture

- To introduce modeling of mechanical systems.
- To introduce modeling of electrical systems.

MODELING OF MECHANICAL SYSTEMS

Definition: *Mass is considered a property of an element that stores the kinetic energy of translational motion.*

$$M = \frac{W}{g} \qquad g = 9.8066 \text{ m/sec}^2$$

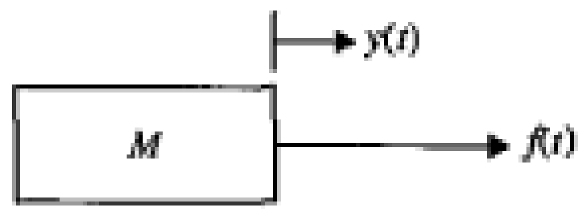
Translational Motion

The motion of translation is defined as a motion that takes place along a straight or curved path. The variables that are used to describe translational motion are **acceleration**, **velocity**, and **displacement.**

MODELING OF MECHANICAL SYSTEMS

Newton's law of motion states that the algebraic sum of external forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction. The law can be expressed as

$$\sum_{external} \text{forces} = Ma$$



where *M* denotes the mass, and *a* is the acceleration in the direction considered.

$$f(t) = Ma(t) = M\frac{d^2y(t)}{dt^2} = M\frac{dv(t)}{dt}$$

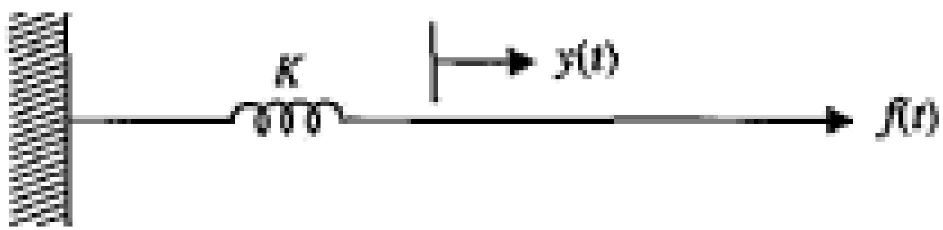
MODELING OF MECHANICAL SYSTEMS

Linear spring.

a spring is considered to be an element that stores potential energy.

$$f(t) = Ky(t)$$

where *K* is the spring constant, or simply stiffness. The model representing a linear spring element is shown in Fig. If the spring is preloaded with a preload tension of *T*, then $f(t) - T = Ky(t)$



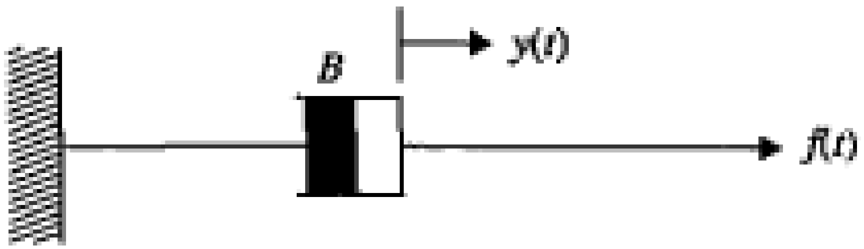
MODELING OF MECHANICAL SYSTEMS

Friction for translation motion.

Whenever there is motion or tendency of motion between two physical elements, frictional forces exist.

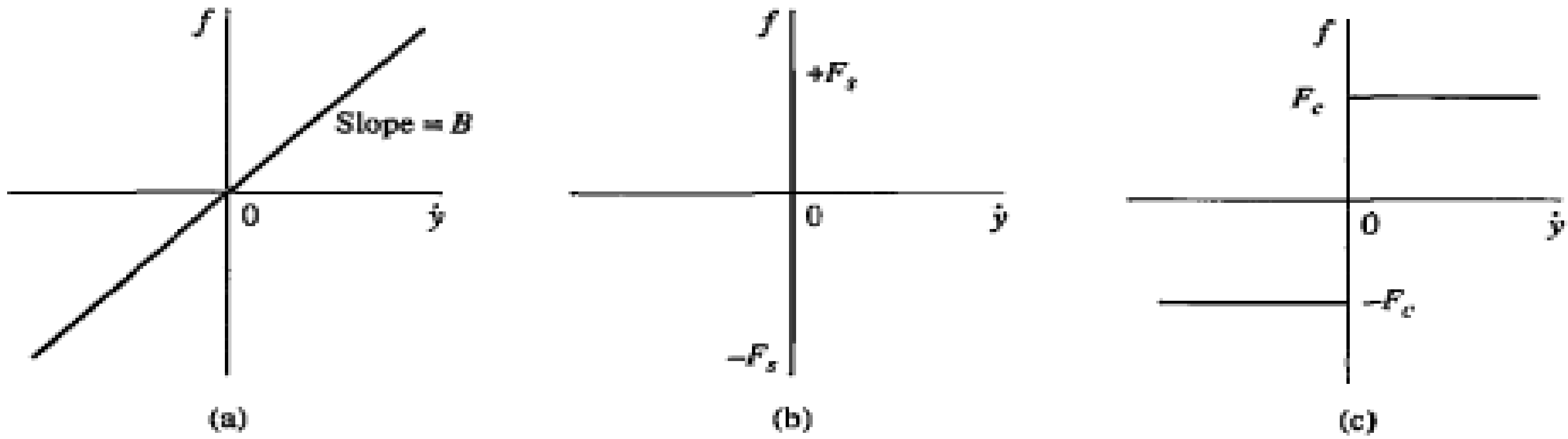
Three different types of friction are commonly used in practical systems: **viscous friction, static friction,** and **Coulomb friction.**

Viscous friction. *Viscous friction represents a retarding force that is a linear relationship between the applied force and velocity.* where *B* is the viscous frictional coefficient



$$f(t) = B\frac{dy(t)}{dt}$$

MODELING OF MECHANICAL SYSTEMS



Static friction. *Static friction represents a retarding force that tends to prevent motion from beginning*

Coulomb friction. *Coulomb friction is a retarding force that has constant amplitude with respect to the change of velocity, but the sign of the frictional force changes with the reversal of the direction of velocity.*

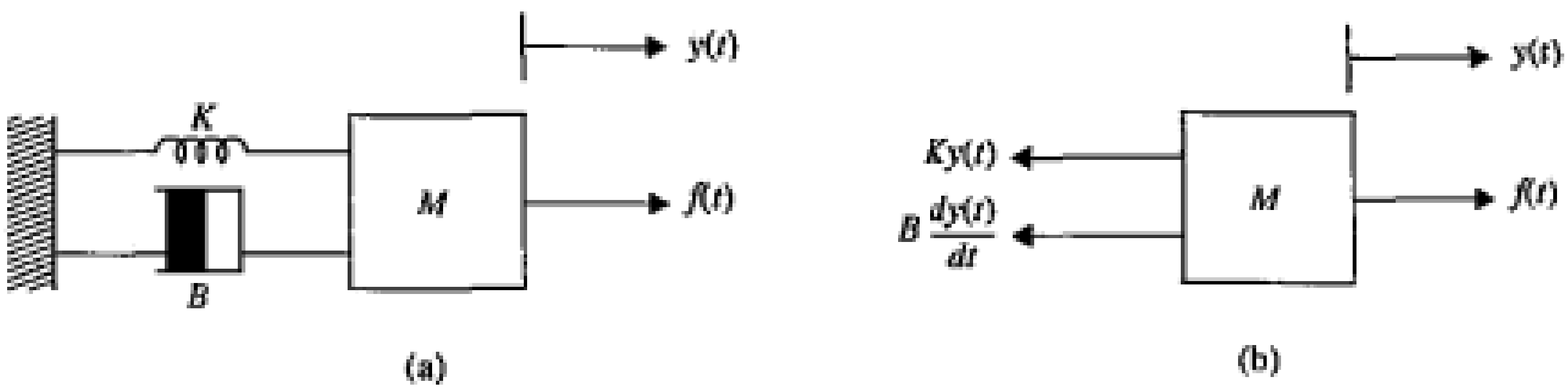
MODELING OF MECHANICAL SYSTEMS

TABLE 4-1 Basic Translational Mechanical System Properties and Their Units

Parameter	Symbol Used	SI Units	Other Units	Conversion Factors
Mass	M	kilogram (kg)	slug ft/sec^2	1 kg = 1000 g = 2.2046 lb(mass) = 35.274 oz(mass) = 0.06852 slug
Distance	y	meter (m)	ft in	1 m = 3.2808 ft = 39.37 in 1 in. = 25.4 mm 1 ft = 0.3048 m
Velocity	v	m/sec	ft/sec in/sec	
Acceleration	a	m/sec^2	ft/sec^2 in/sec^2	
Force	f	Newton (N)	pound (lb force) dyne	1 N = 0.2248 lb(force) = 3.5969 oz(force) 1 N = 1 kg−m/s^2 1 dyn = 1 g−cm/s^2
Spring Constant	K	N/m	lb/ft	
Viscous Friction Coefficient	B	N/m/sec	lb/ft/sec	

EXAMPLE (1)

Consider the mass-spring-friction system shown in Fig. 1-a The linear motion concerned is in the horizontal direction. The free-body diagram of the system is shown in Fig. 1-b). The force equation of the system is



$$f(t) - B\frac{dy(t)}{dt} - Ky(t) = M\frac{d^2y(t)}{dt^2}$$

$$\frac{d^2y(t)}{dt^2} = -\frac{B}{M}\frac{dy(t)}{dt} - \frac{K}{M}y(t) + \frac{1}{M}f(t)$$

$$\dot{y}(t) = \left(\frac{dy(t)}{dt}\right) \text{ and } \ddot{y}(t) = \left(\frac{d^2y(t)}{dt^2}\right)$$

represent velocity and acceleration, respectively

$$\ddot{y}(t) + \frac{B}{M}\dot{y}(t) + \frac{K}{M}y(t) = \frac{1}{M}f(t)$$

where $y(t)$ is the output and $\frac{f(t)}{M}$ is considered the input.

For zero initial conditions, the transfer function between *Y(s)* and *F(s)* is obtained by taking the Laplace transform on both sides

$$\frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

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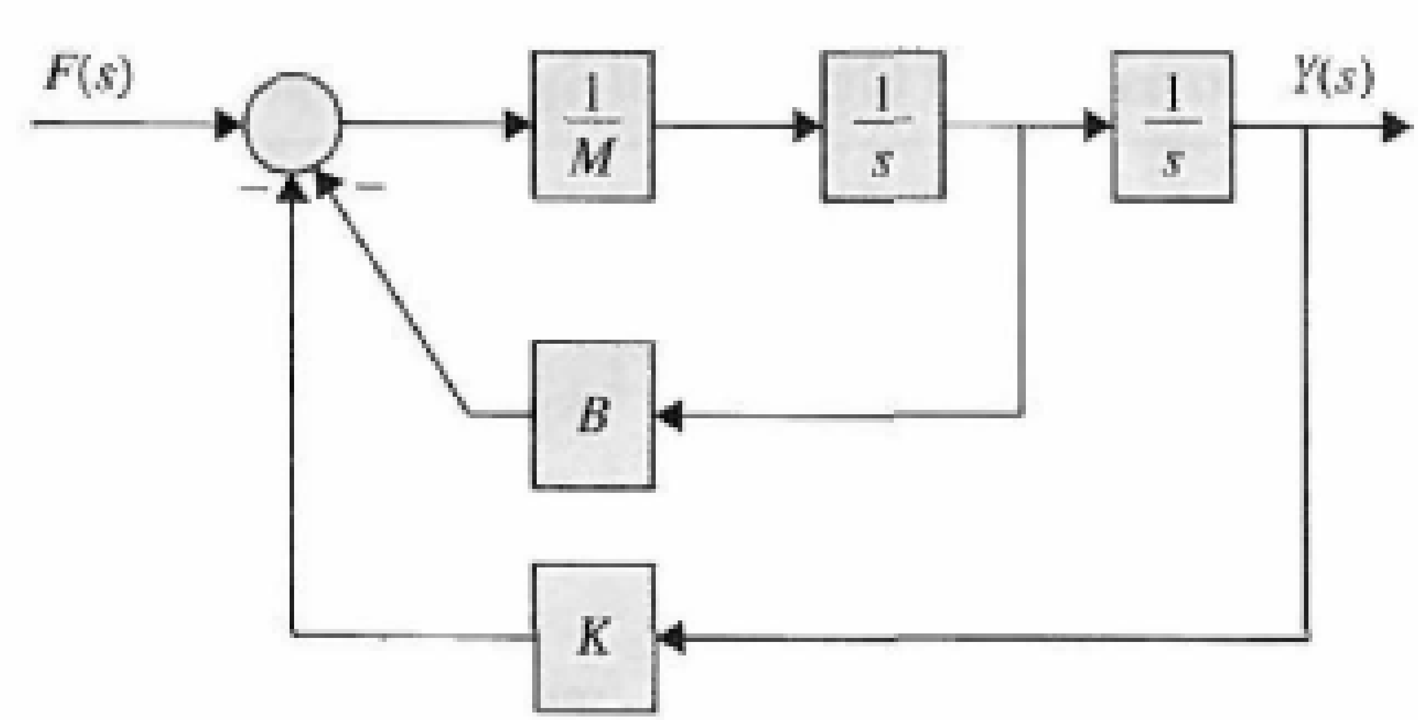
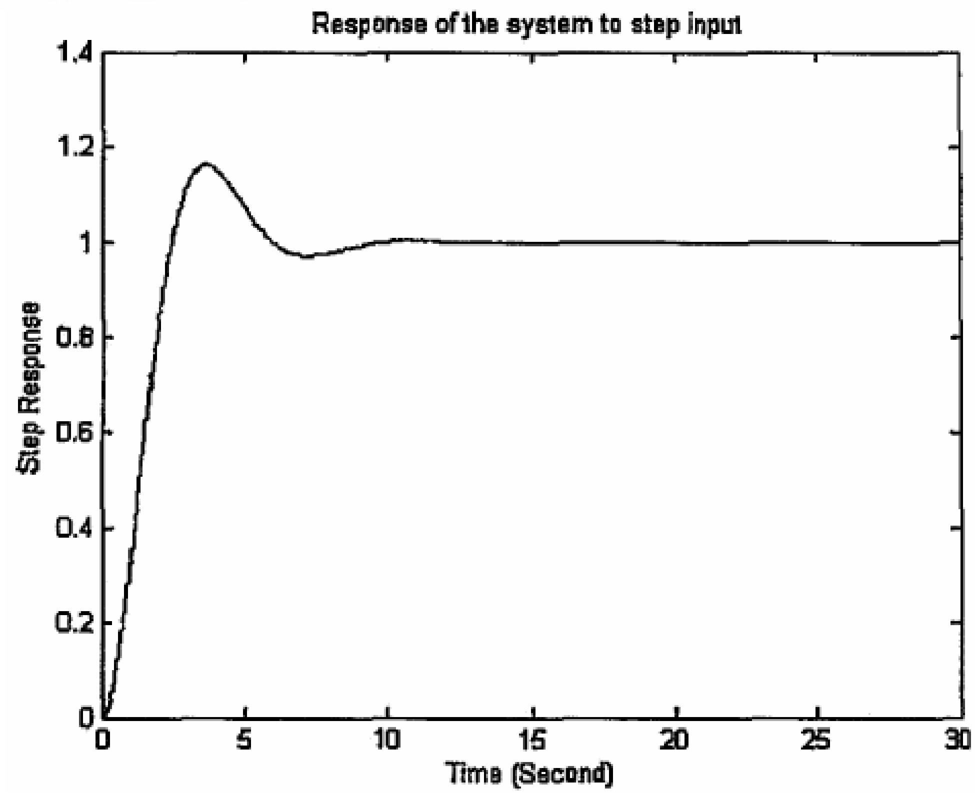


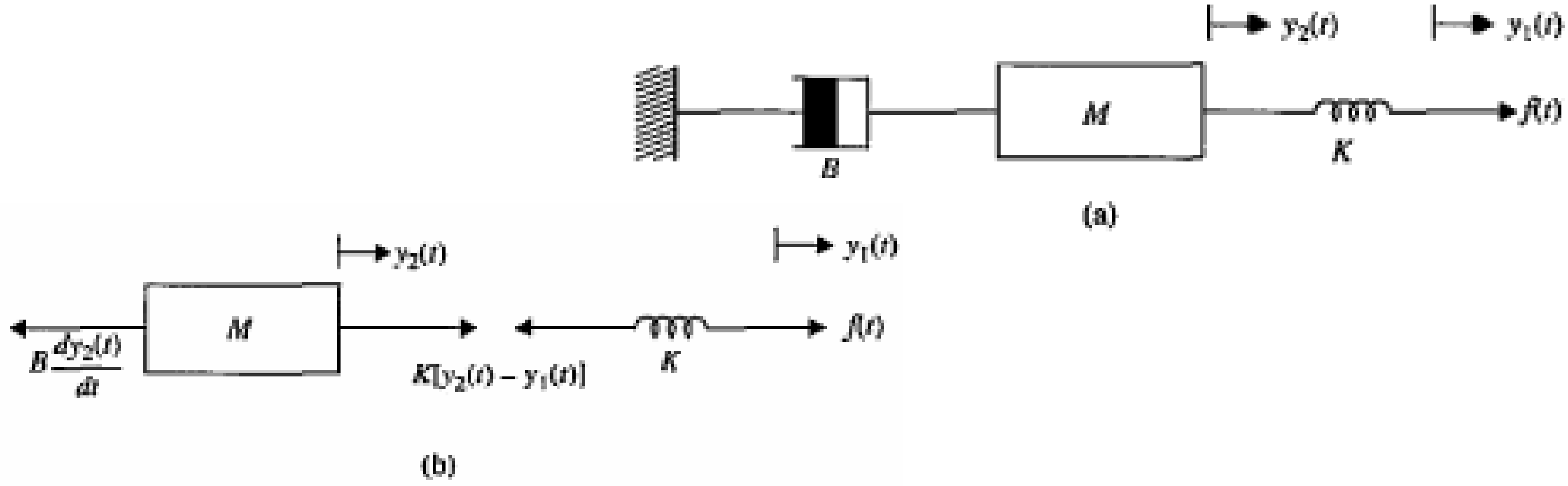
Time domain step response for Eq. (4-12) is calculated using MATLAB for K = 1, M = 1, B = 1:

```
K=1; M=1; B=1;
t=0:0.02:30;
num = [1];
den = [M B K];
G = tf(num,den);
y1 = STEP (G,t);
plot(t, y1);
xlabel('Time (Second)');ylabel('Step Response')
title('Response of the system to step input')
```



EXAMPLE (2)

consider the system shown in Fig



$$f(t) = K[y_1(t) - y_2(t)]$$

$$-K[y_2(t) - y_1(t)] - B\frac{dy_2(t)}{dt} = M\frac{d^2y_2(t)}{dt^2}$$

These equations are rearranged in input-output form as

$$\frac{d^2y_2(t)}{dt^2} + \frac{B}{M}\frac{dy_2(t)}{dt} + \frac{K}{M}y_2(t) = \frac{K}{M}y_1(t)$$

For zero initial conditions, the transfer function between *Y\(s)* and *Y2(s)* is obtained by taking the Laplace transform on both sides

$$\frac{Y_2(s)}{Y_1(s)} = \frac{K}{Ms^2 + Bs + K}$$

Rotational Motion

The rotational motion of a body can be defined as motion about a fixed axis. The extension of Newton's law of motion for rotational motion states that the *algebraic sum of moments or torque about a fixed axis is equal to the product of the inertia and the angular acceleration about the axis.* Or

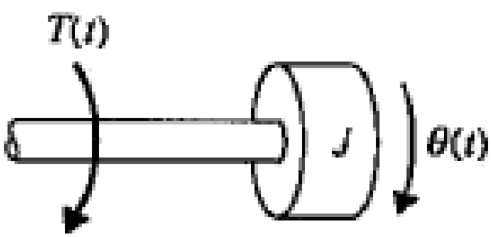
$\sum \text{torques} = J\alpha$ where *J* denotes the inertia and *a* is the angular acceleration.

The other variables generally used to describe the motion of rotation are **torque** T, **angular velocity** ω, and **angular displacement** θ. The elements involved with the rotational motion

Inertia. *Inertia, J,*

The inertia of a given element depends on the geometric composition about the axis of rotation and its density

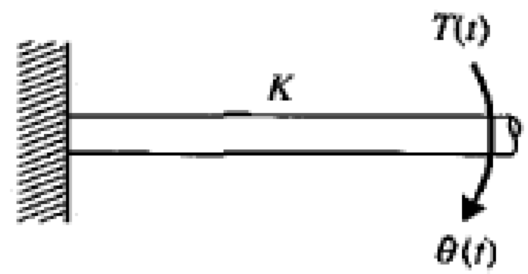
$$J = \frac{1}{2}Mr^2$$



$$T(t) = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$

Torsional spring.

As with the linear spring for translational motion, a **torsional** spring **constant *K,*** in torque-per-unit angular displacement, can be devised to represent the compliance of a rod or a shaft when it is subject to an applied torque.



$$T(t) = K\theta(t)$$

Friction for rotational motion.

The three types of friction described for translational motion can be carried over to the motion of rotation.

- **Viscous friction.**

$$T(t) = B\frac{d\theta(t)}{dt}$$

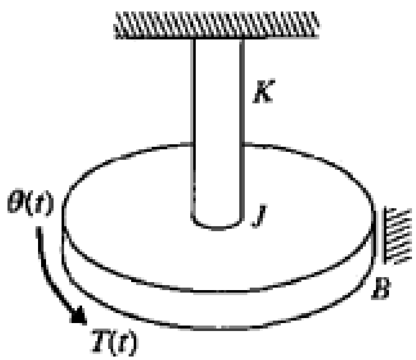
Static friction.

$$T(t) = \pm(F_s)|_{\dot{\theta}=0}$$

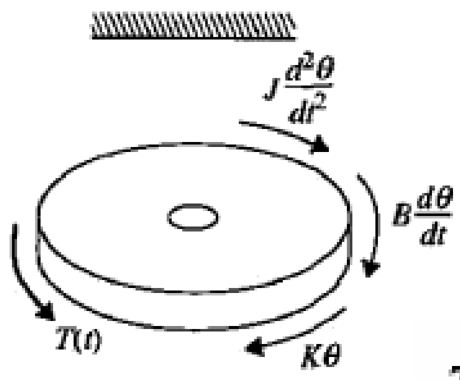
Coulomb friction.

$$T(t) = F_c \frac{\frac{d\theta(t)}{dt}}{\left|\frac{d\theta(t)}{dt}\right|}$$

EXAMPLE (3)



(a)



(b)

$$T(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} + K\theta(t)$$

EXAMPLE 4

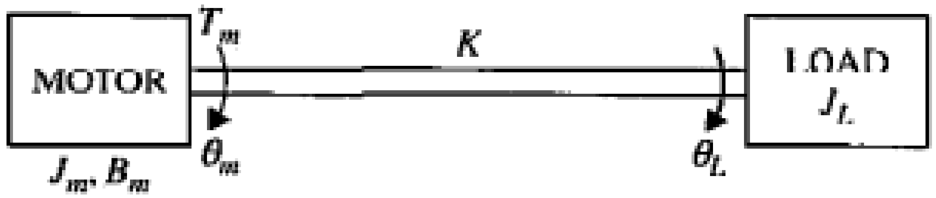


Fig. shows the diagram of a motor coupled to an inertial load through a shaft with a spring constant *K. A* non-rigid coupling between two mechanical components in a control system often causes torsional resonances that can be transmitted to all parts of the system. The system variables and parameters are defined as follows:

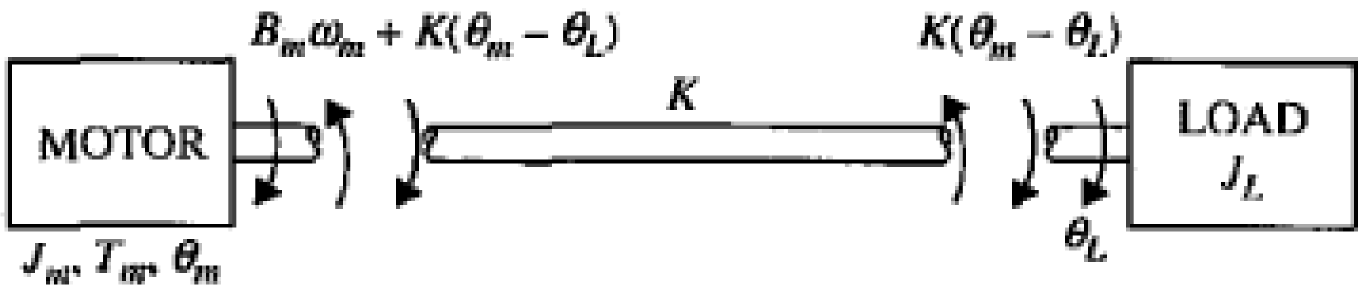
$T_m(t)$ = motor torque

B_m = motor viscous-friction coefficient

K = spring constant of the shaft

$\theta_m(t)$ = motor displacement

$\omega_m(t)$ = motor velocity



$$\frac{d^2\theta_m(t)}{dt^2} = -\frac{B_m}{J_m}\frac{d\theta_m(t)}{dt} - \frac{K}{J_m}[\theta_m(t) - \theta_L(t)] + \frac{1}{J_m}T_m(t)$$

J_m = motor inertia

$\theta_L(t)$ = load displacement

$\omega_L(t)$ = load velocity

J_L = load inertia

$$K[\theta_m(t) - \theta_L(t)] = J_L\frac{d^2\theta_L(t)}{dt^2}$$

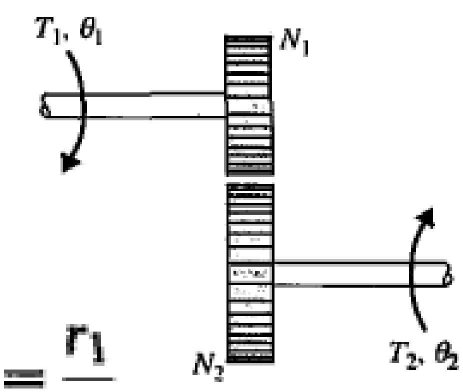
Gear Trains

$$r_1 N_2 = r_2 N_1$$

$$\theta_1 r_1 = \theta_2 r_2$$

$$T_1\theta_1 = T_2\theta_2$$

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$$



MODELING OF SIMPLE ELECTRICAL SYSTEMS

Modeling of Passive Electrical Elements

Resistors:

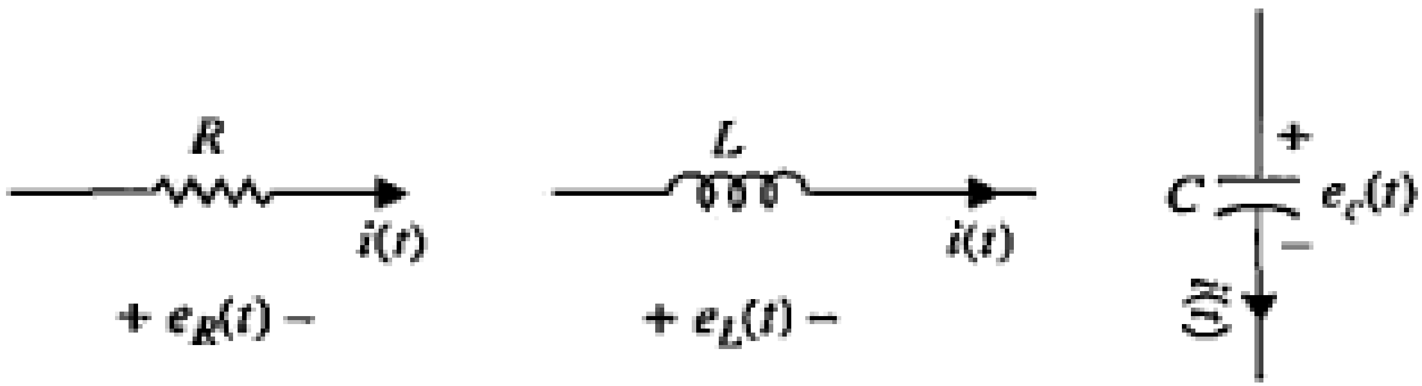
$$e_R(t) = i(t)R$$

Inductors:

$$e_L(t) = L\frac{di(t)}{dt}$$

Capacitor:

$$e_c(t) = \int \frac{i(t)}{C}dt$$



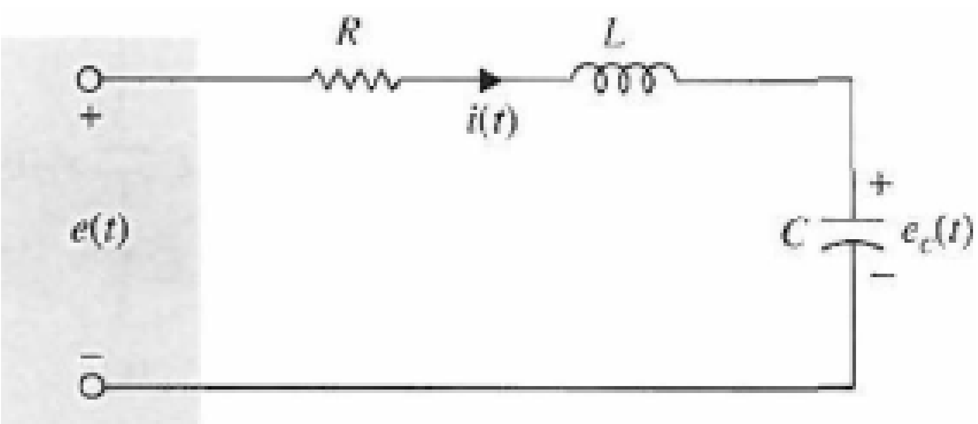
(a) (b) (c)

Modeling of Electrical Networks

Current Law or Loop Method: The algebraic summation of all currents entering a node is zero.

Voltage Law or Node Method: The algebraic sum of all voltage drops around a complete closed loop is zero.

EXAMPLE(4)



$$e(t) = e_R + e_L + e_c$$

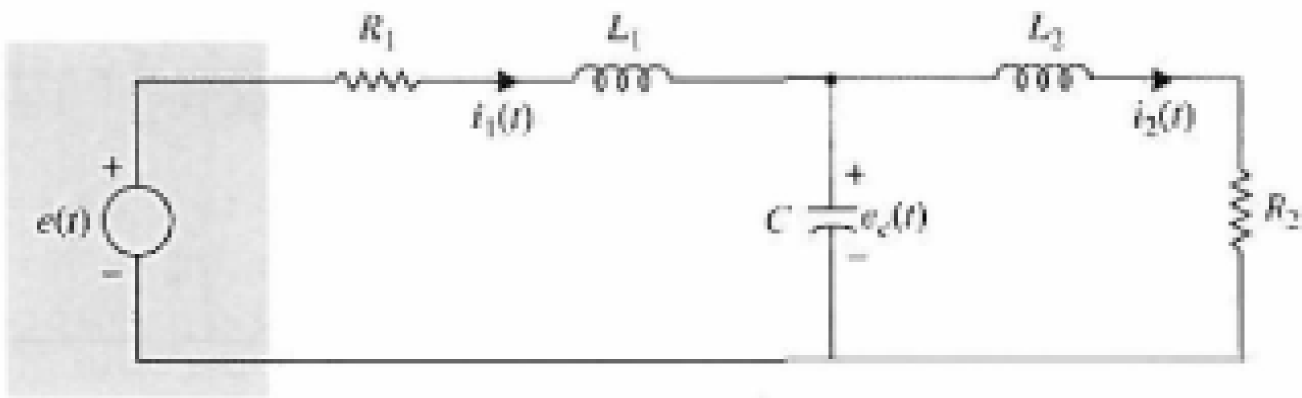
$$e(t) = +e_c(t) + Ri(t) + L\frac{di(t)}{dt}$$

$$C\frac{de_c(t)}{dt} = i(t)$$

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{i(t)}{C} = \frac{de(t)}{dt}$$

$$\frac{I(s)}{E(s)} = \frac{(1/L)s^{-1}}{1 + (R/L)s^{-1} + (1/LC)s^{-2}} = \frac{Cs}{1 + RCs + LCs^2}$$

EXAMPLE (5)



(a)

$$L_1 \frac{di_1(t)}{dt} = -R_1 i_1(t) - e_c(t) + e(t)$$

$$L_2 \frac{di_2(t)}{dt} = -R_2 i_2(t) + e_c(t)$$

$$C \frac{de_c(t)}{dt} = i_1(t) - i_2(t)$$

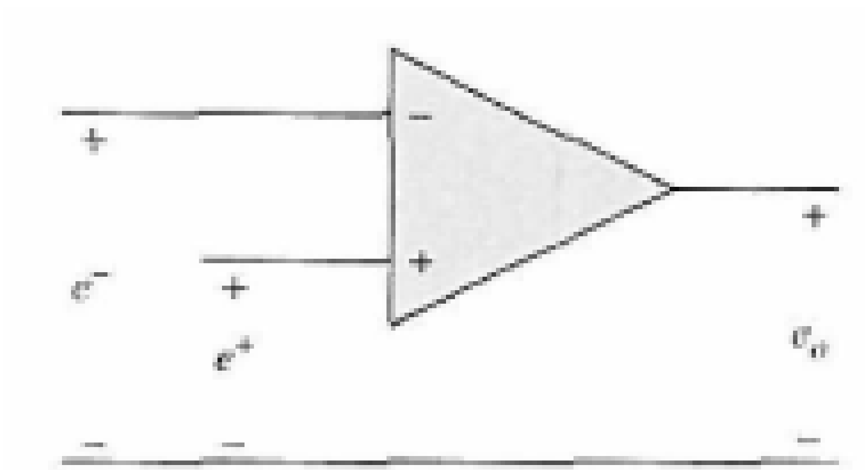
$$\frac{I_1(s)}{E(s)}$$

$$\frac{I_2(s)}{E(s)}$$

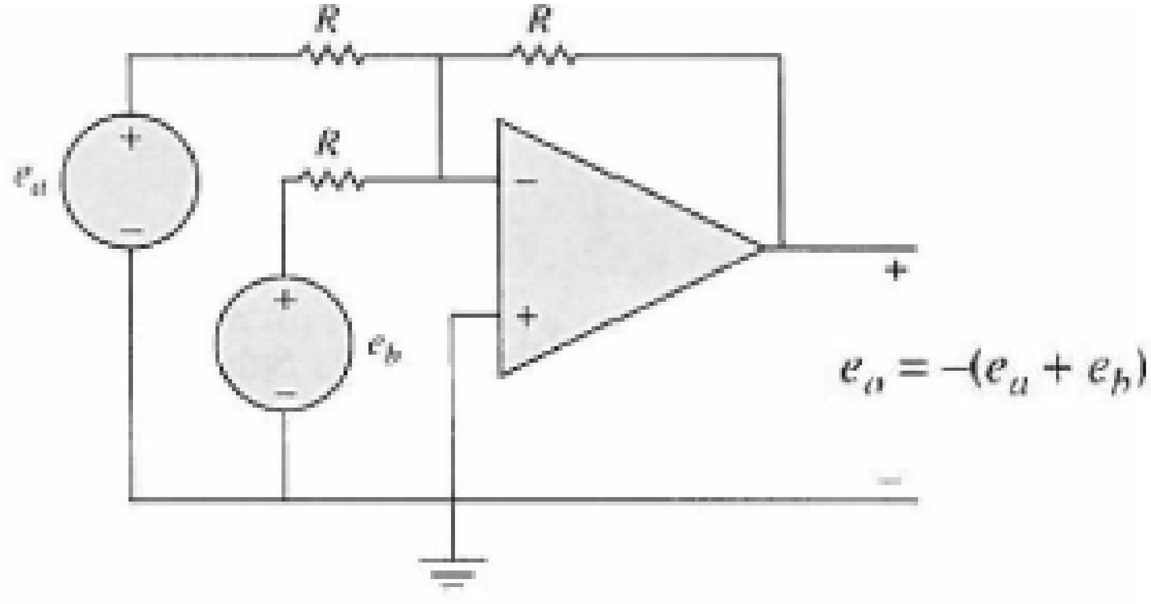
$$\frac{E_c(s)}{E(s)}$$

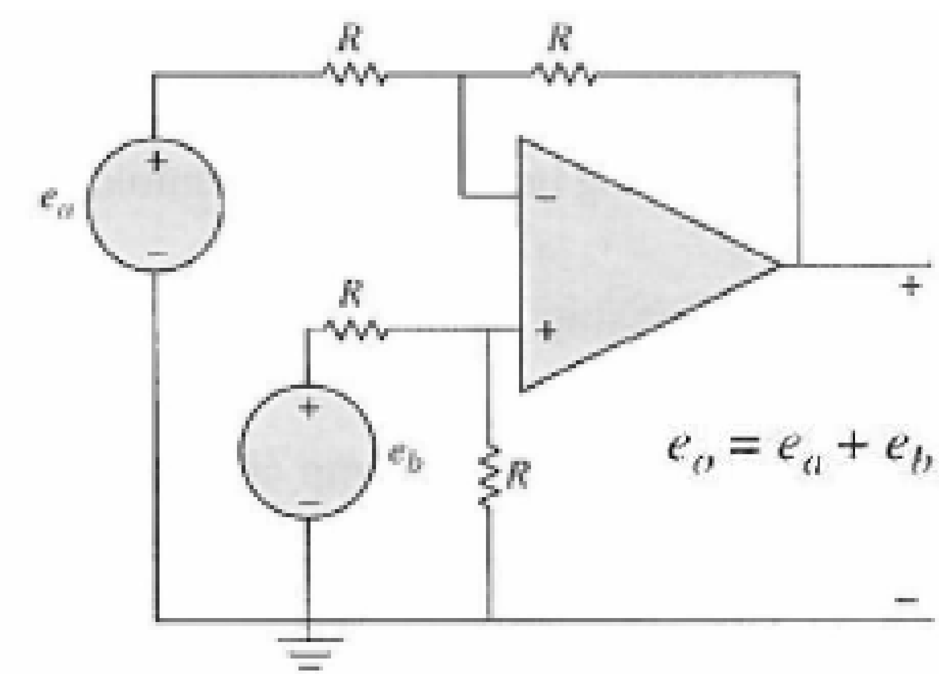
MODELING OF ACTIVE ELECTRICAL ELEMENTS: OPERATIONAL AMPLIFIERS

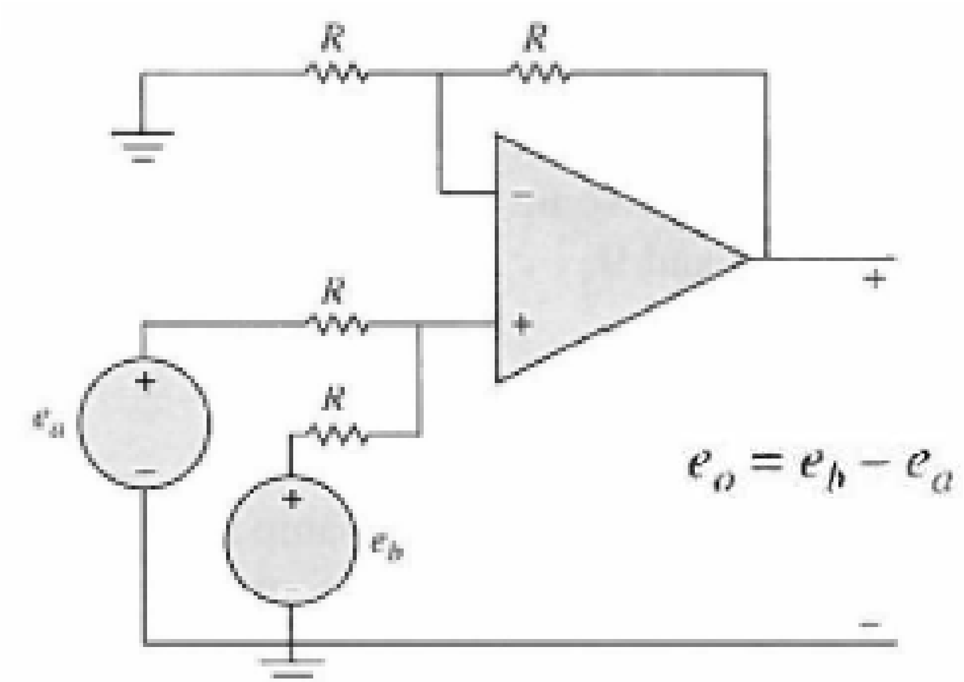
Operational amplifiers, or simply **op-amps,** offer a convenient way to build, implement, or realize continuous-data or s-domain transfer functions. In control systems, op-amps are often used to implement the controllers or compensators that evolve from the control system design process, so in this section we illustrate common op-amp configurations.

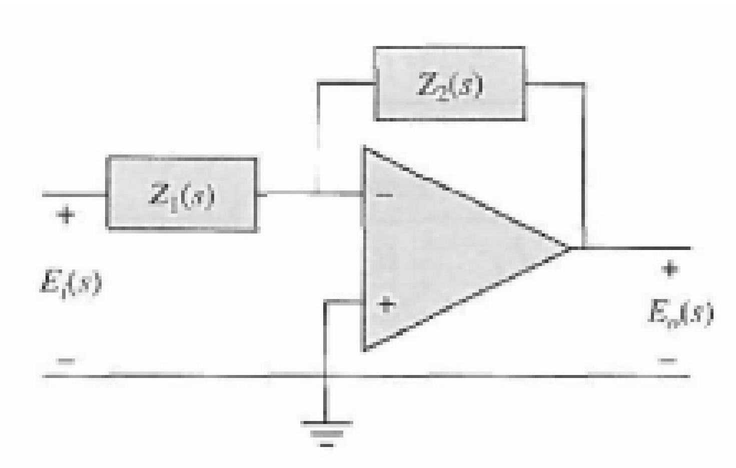


Sums and Differences







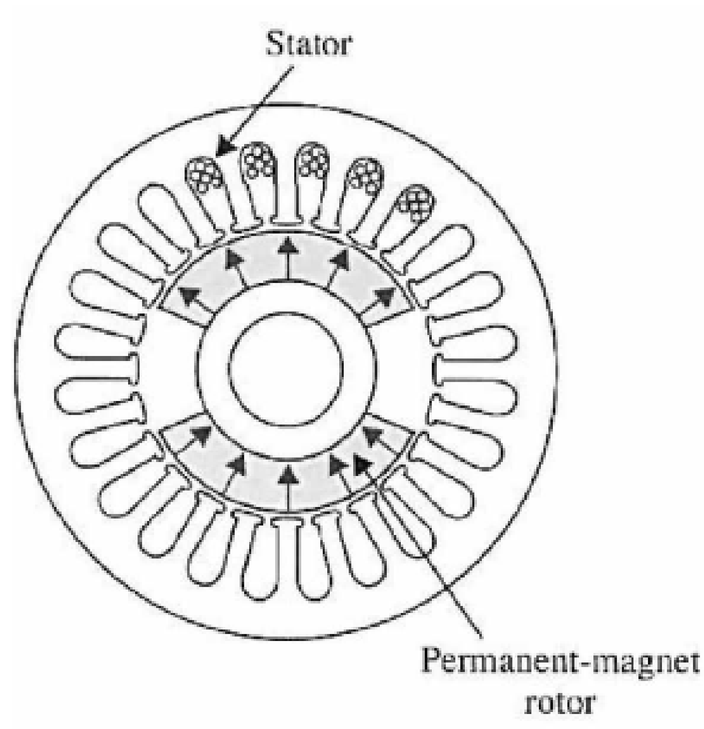


$$G(s) = \frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = \frac{-1}{Z_1(s)Y_2(s)}$$

$$= -Z_2(s)Y_1(s) = -\frac{Y_1(s)}{Y_2(s)}$$

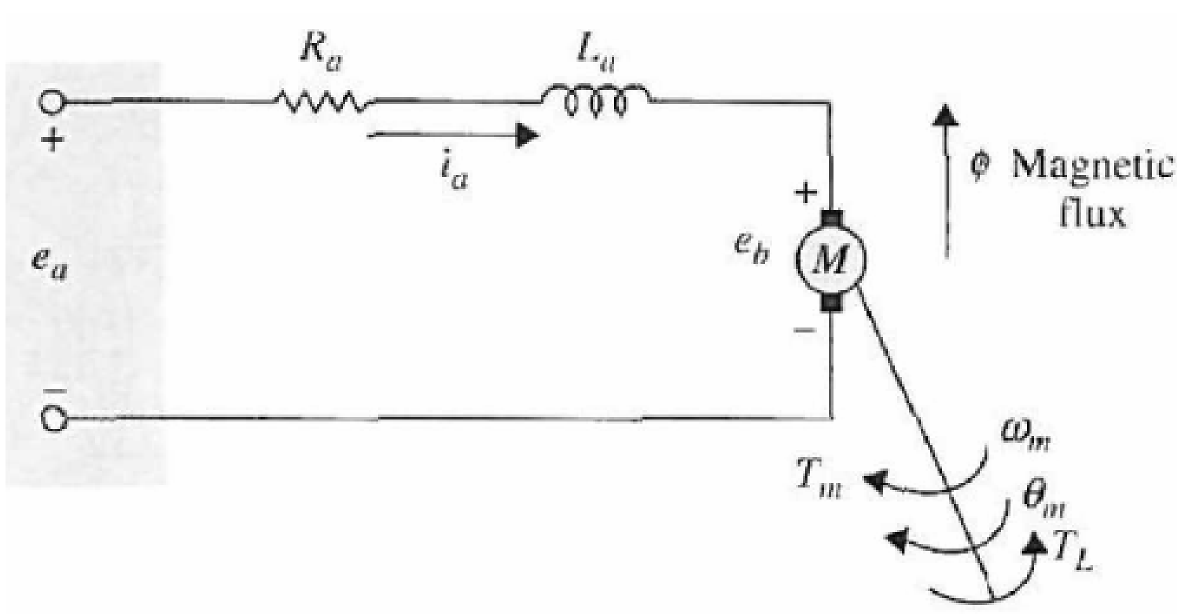
Mathematical Modeling of PM DC Motors

Dc motors are extensively used in control systems. In this section we establish the mathematical model for dc motors. As it will be demonstrated here, the mathematical model of a dc motor is linear



Mathematical Modeling of PM DC Motors

The armature is modeled as a circuit with resistance *Ra* connected in series with an inductance *La,* and a voltage source *eb,* representing the back emf electromotive force) in the armature when the rotor rotates



Mathematical Modeling of PM DC Motors

The motor variables and parameters are defined as follows:

$i_a(t)$ = armature current

R_a = armature resistance

$e_b(t)$ = back emf

$T_L(t)$ = load torque

$T_m(t)$ = motor torque

$\theta_m(t)$ = rotor displacement

K_i = torque constant

L_a = armature inductance

$e_a(t)$ = applied voltage

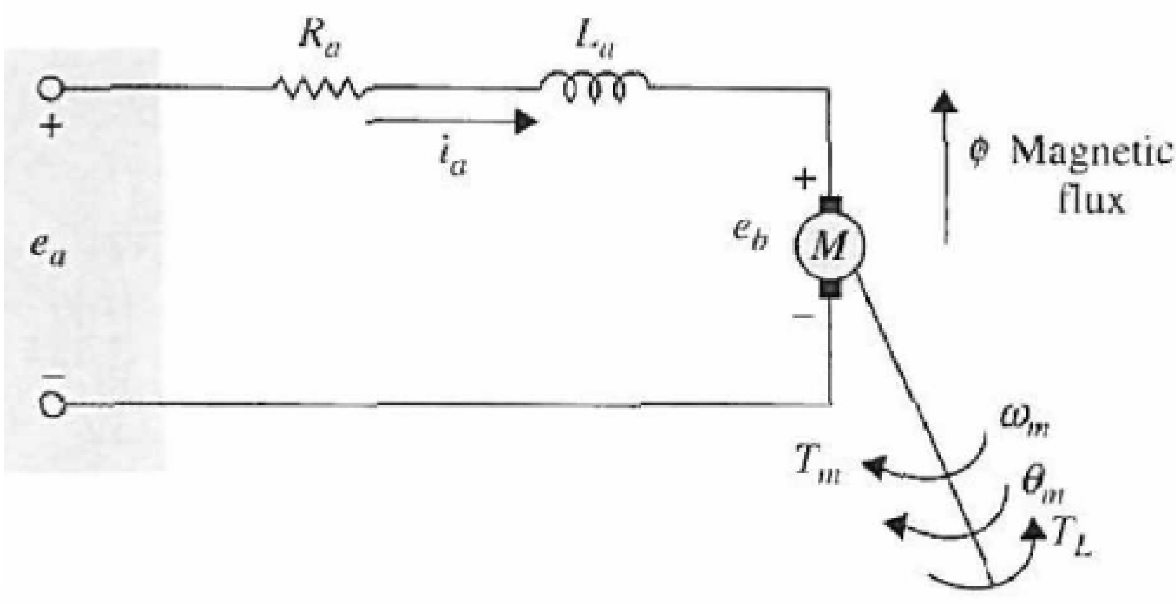
K_b = back-emf constant

ϕ = magnetic flux in the air gap

$\omega_m(t)$ = rotor angular velocity

J_m = rotor inertia

B_m = viscous-friction coefficient



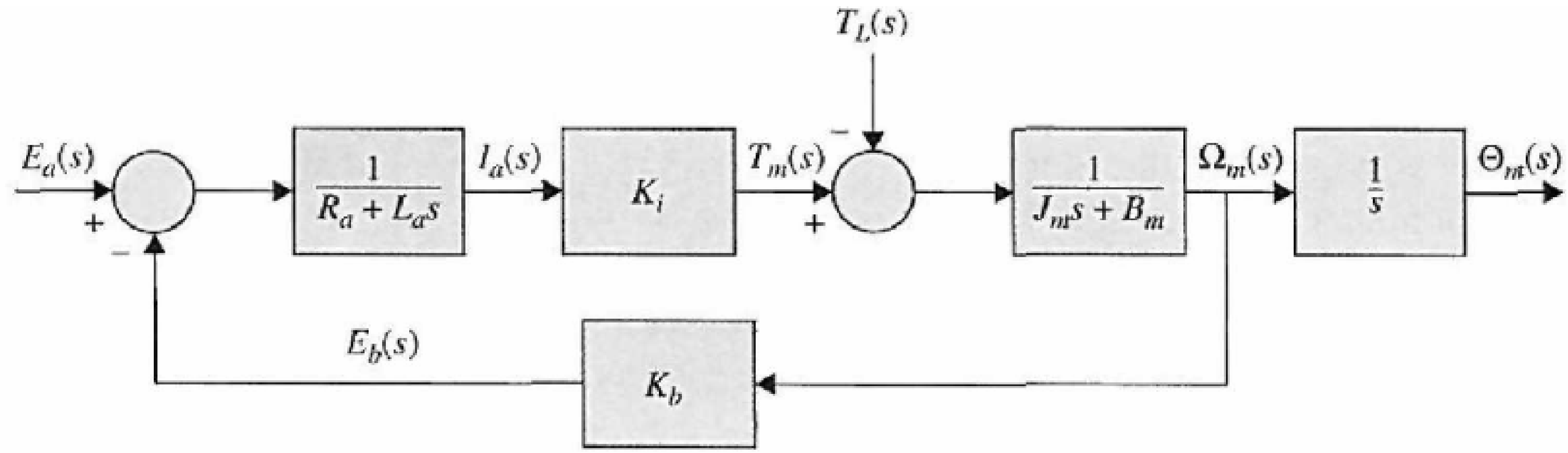
$$T_m(t) = K_m(t)\phi i_a(t) \Longrightarrow T_m(t) = K_i i_a(t)$$

$$\frac{di_a(t)}{dt} = \frac{1}{L_a}e_a(t) - \frac{R_a}{L_a}i_a(t) - \frac{1}{L_a}e_b(t)$$

$$e_b(t) = K_b\frac{d\theta_m(t)}{dt} = K_b\omega_m(t)$$

$$\frac{d^2\theta_m(t)}{dt^2} = \frac{1}{J_m}T_m(t) - \frac{1}{J_m}T_L(t) - \frac{B_m}{J_m}\frac{d\theta_m(t)}{dt}$$

where 7/,(0 represents a load frictional torque such as Coulomb friction.



$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_i}{L_a J_m s^3 + (R_a J_m + B_m L_a)s^2 + (K_b K_i + R_a B_m)s}$$