Control System II
EE 412

Lecture 2

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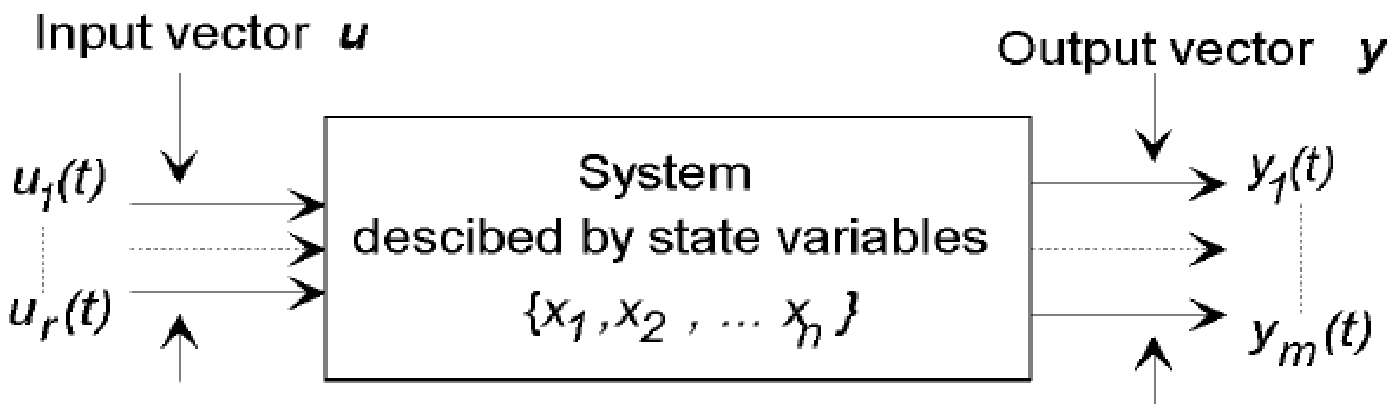
State Space Representation

- The concept of the state of a dynamic system refers to a minimum set of variables, known as state variables, that fully describe the system and its response to any given set of inputs



The state variables are an *internal* description of the system which completely characterize the system state at any time *t*, and from which any output variables *yi*(*t*) may be computed.

State Space Representation

The complete system model for a linear time-invariant system consists of:

(i) a set of *n* state equations, defined in terms of the matrices **A** and **B**, and
(ii) a set of output equations that relate any output variables of interest to the state variables and inputs, and expressed in terms of the **C** and **D** matrices.

The task of modeling the system is to derive the elements of the matrices, and to write the system model in the form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \end{aligned}$$

The matrices **A** and **B** are properties of the system and are determined by the system structure and elements. The output equation matrices **C** and **D** are determined by the particular choice of output variables.

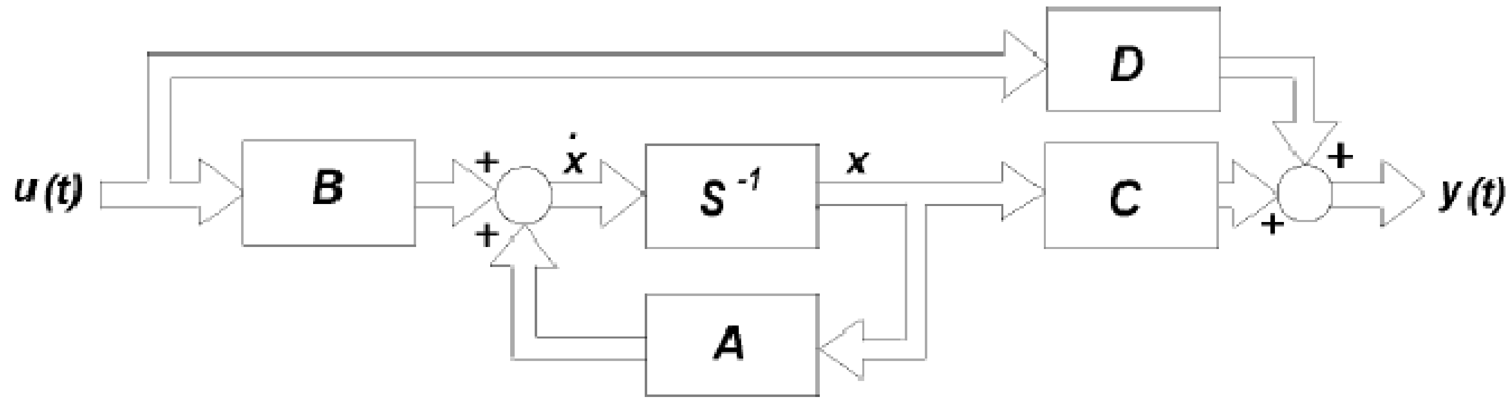
Block Diagram Representation of Linear Systems Described by State Equations

Step 1: Draw *n* integrator (S−1) blocks, and assign a state variable to the output of each block.

Step 2: At the input to each block (which represents the derivative of its state variable) draw a summing element.

Step 3: Use the state equations to connect the state variables and inputs to the summing elements through scaling operator blocks.

Step 4: Expand the output equations and sum the state variables and inputs through a set of scaling operators to form the components of the output.



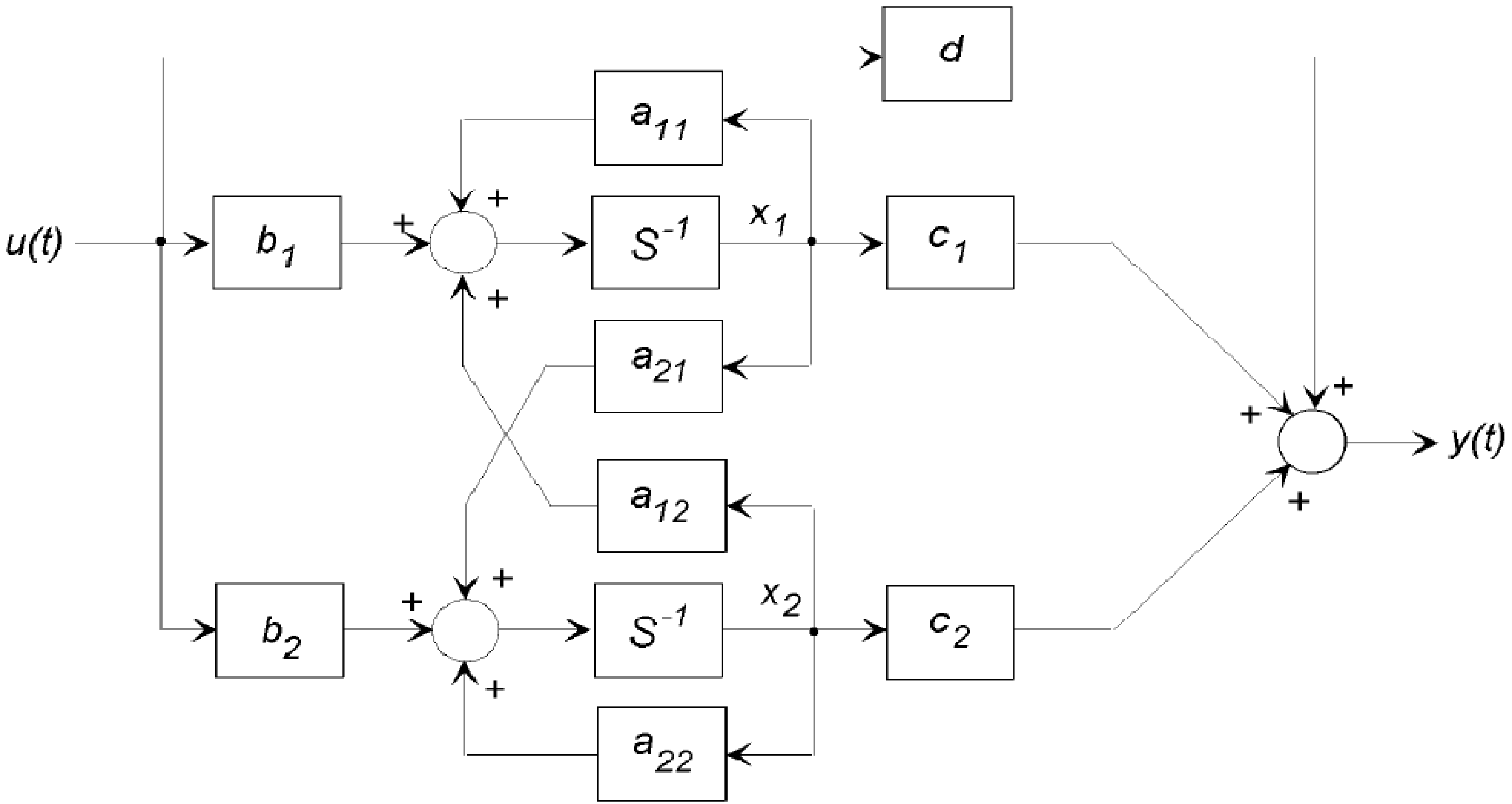
Example 1

Draw a block diagram for the general second-order, single-input single-output system:

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + du(t).
\end{aligned}
$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + du(t).$$



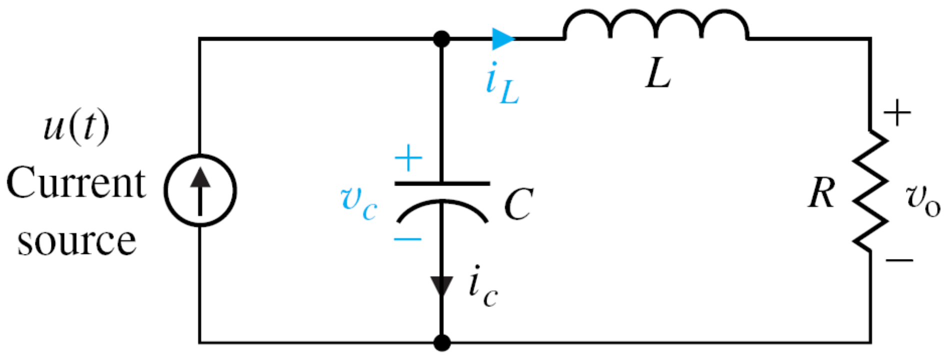
The overall modeling procedure developed in this chapter is based on the following steps:

1. Determination of the system order n and selection of a set of state variables from the linear graph system representation.

2. Generation of a set of state equations and the system **A** and **B** matrices using a well defined methodology. This step is also based on the linear graph system description.

3. Determination of a suitable set of output equations and derivation of the appropriate **C** and **D** matrices.

Consider the following RLC circuit



We can choose state variables to be $x_1 = v_c(t),\ x_2 = i_L(t),$

Alternatively, we may choose $\hat{x}_1 = v_c(t),\ \hat{x}_2 = v_L(t).$

This will yield two different sets of state space equations, but both of them have the identical input-output relationship, expressed by

$$\frac{V_0(s)}{U(s)} = \frac{R}{LCs^2 + RCs + 1}.$$

Can you derive this TF?

Linking state space representation and transfer function

- Given a transfer function, there exist infinitely many input-output equivalent state space models.
- We are interested in special formats of state space representation, known as *canonical forms*.
- It is useful to develop a graphical model that relates the state space representation to the corresponding transfer function. The graphical model can be constructed in the form of signal-flow graph or block diagram.

We recall Mason's gain formula when all feedback loops are touching and also touch all forward paths,

$$T = \frac{\sum_k P_k \Delta_k}{\Delta} = \frac{\sum_k P_k}{1 - \sum_{q=1}^{N} L_q} = \frac{\text{Sum of forward path gain}}{1 - \text{sum of feedback loop gain}}$$

Consider a 4th-order TF

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

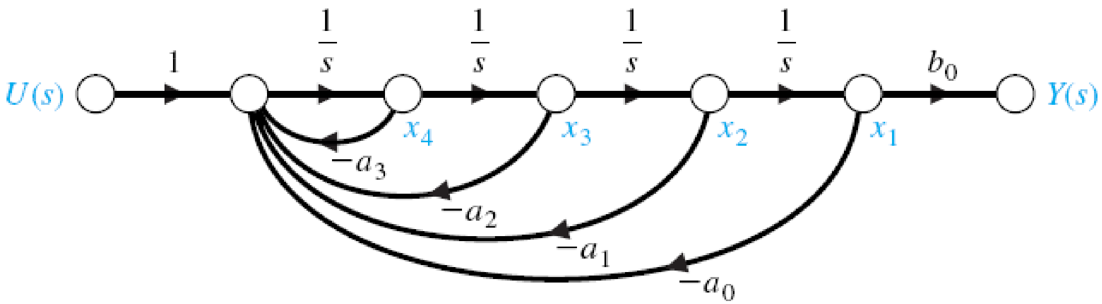
$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

We notice the similarity between this TF and Mason's gain formula above. To represent the system, we use 4 state variables Why?

Signal-flow graph model

This 4th-order system can be represented by

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0 s^{-4}}{1+a_3 s^{-1}+a_2 s^{-2}+a_1 s^{-3}+a_0 s^{-4}}$$



How do you verify this signal-flow graph by Mason's gain formula?

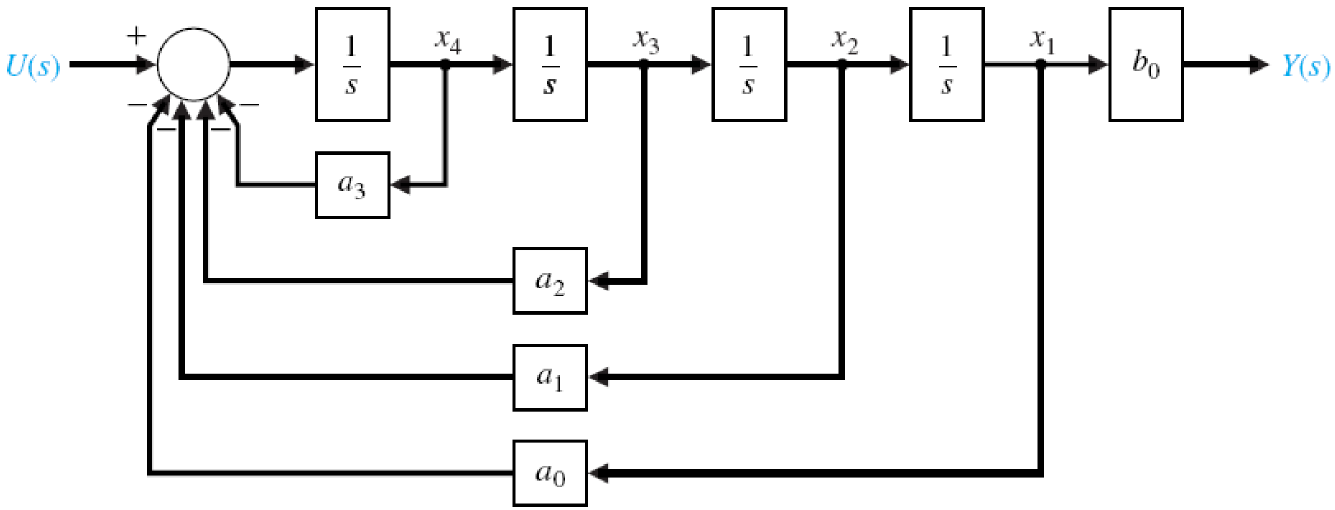
Block diagram model

Again, this 4th-order TF

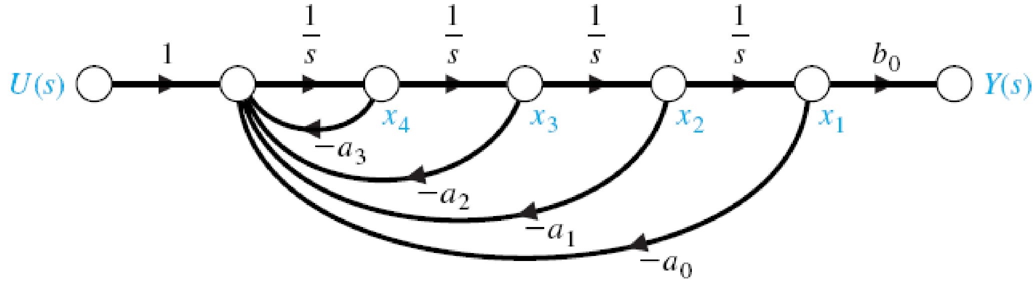
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

can be represented by the block diagram as shown



With either the signal-flow graph or block diagram of the previous 4th-order system,



we define state variables as $x_1 = {}^{y}\!/\!_{b_0},\ x_2 = \dot{x}_1,\ x_3 = \dot{x}_2,\ x_4 = \dot{x}_3,$

then the state space representation is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u$$
$$y = b_0 x_1$$

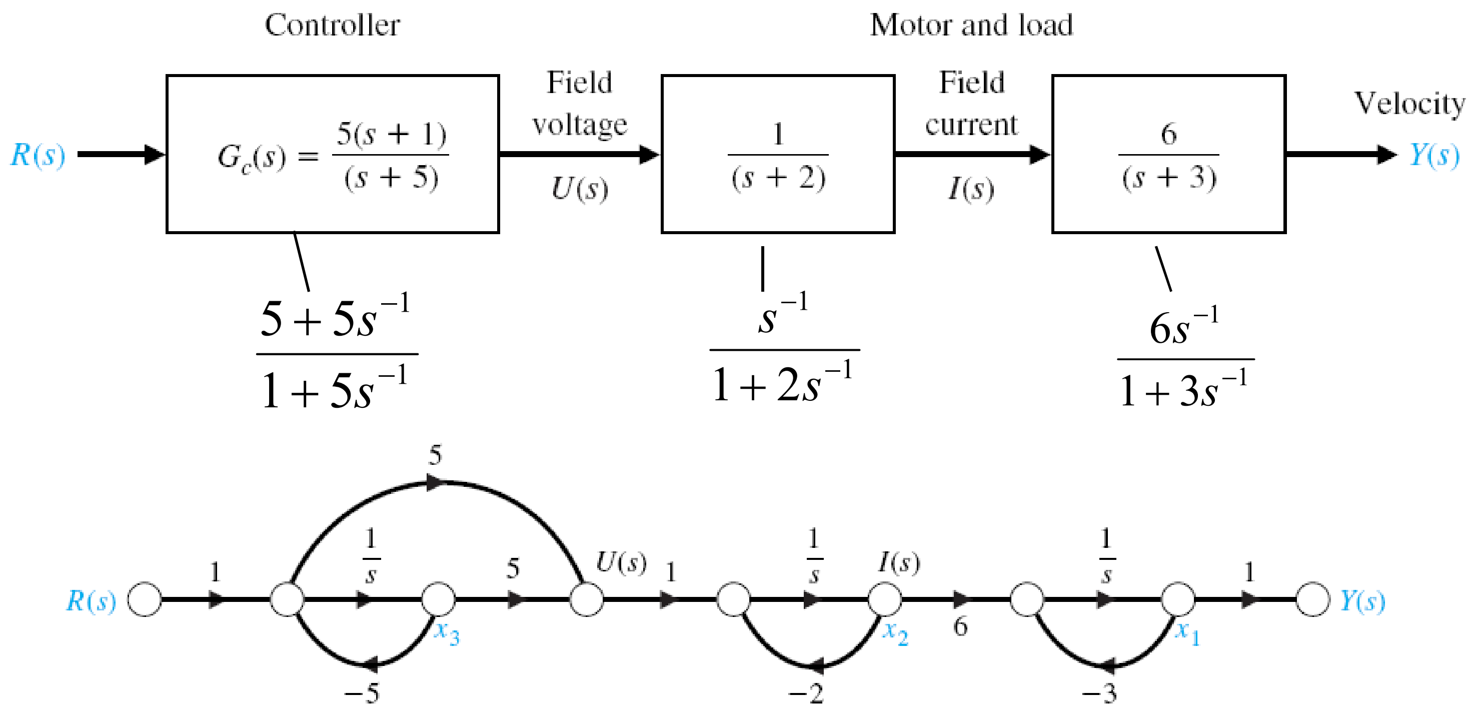
Writing in matrix form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

we have

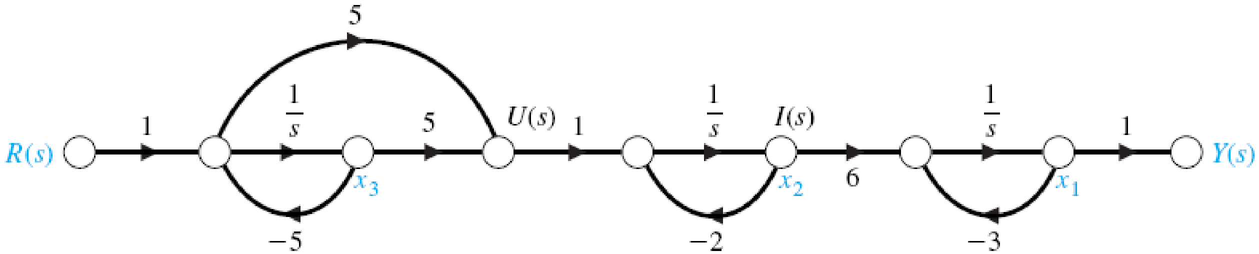
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} b_0 & 0 & 0 & 0 \end{bmatrix}, \qquad D = 0$$

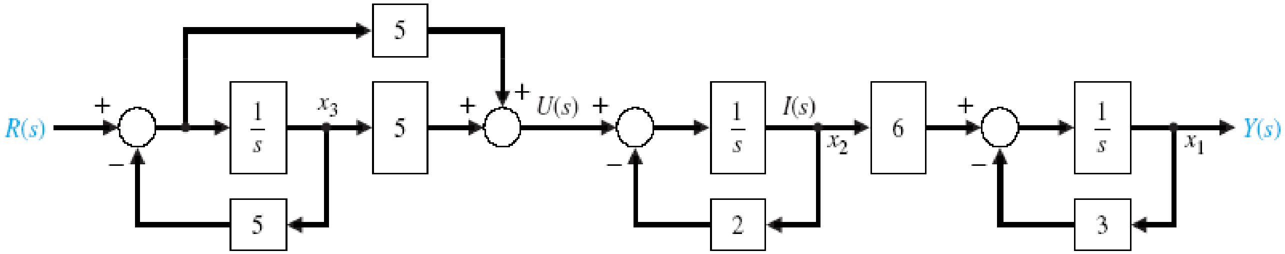
When studying an actual control system block diagram, we wish to select the physical variables as state variables. For example, the block diagram of an open loop DC motor is



We draw the signal-flow diagraph of each block separately and then connect them. We select $x_1 = y(t)$, $x_2 = i(t)$ and $x_3 = (1/4)r(t) - (1/20)u(t)$ to form the state space representation.

Physical state variable model



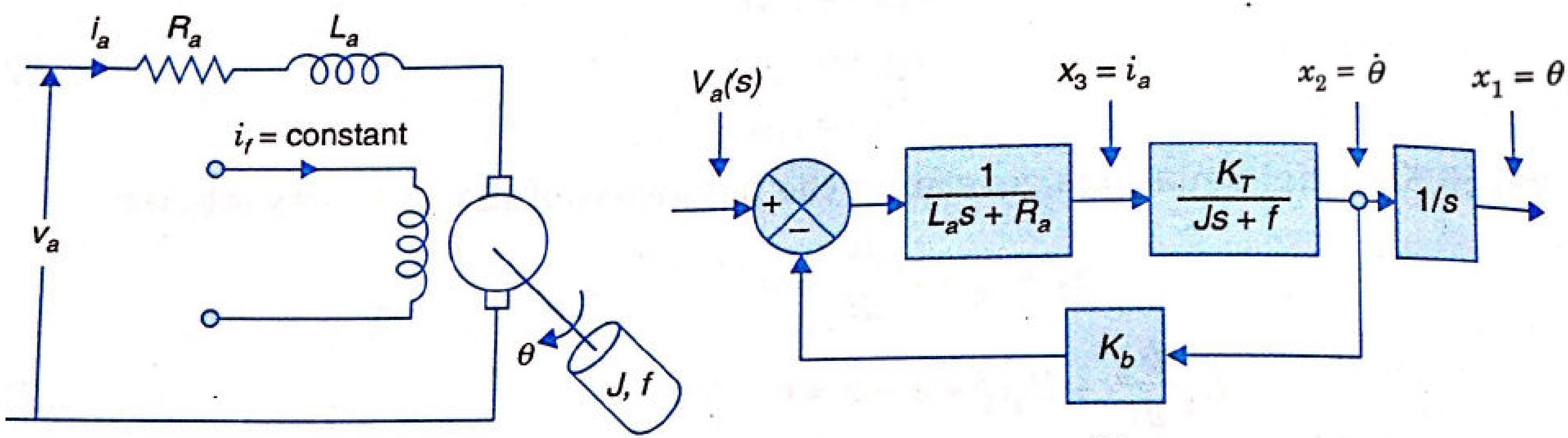


The corresponding state space equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t)$$

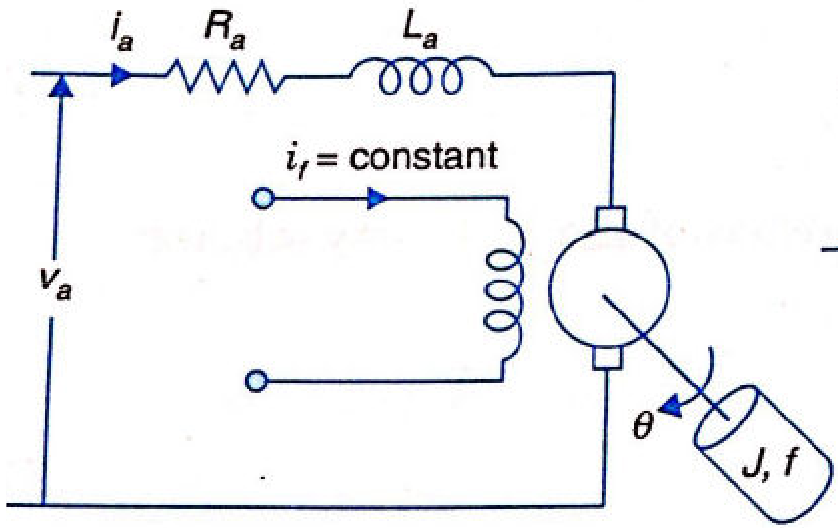
$$y = [1 \quad 0 \quad 0]\mathbf{x}$$

Electro Mechanical System

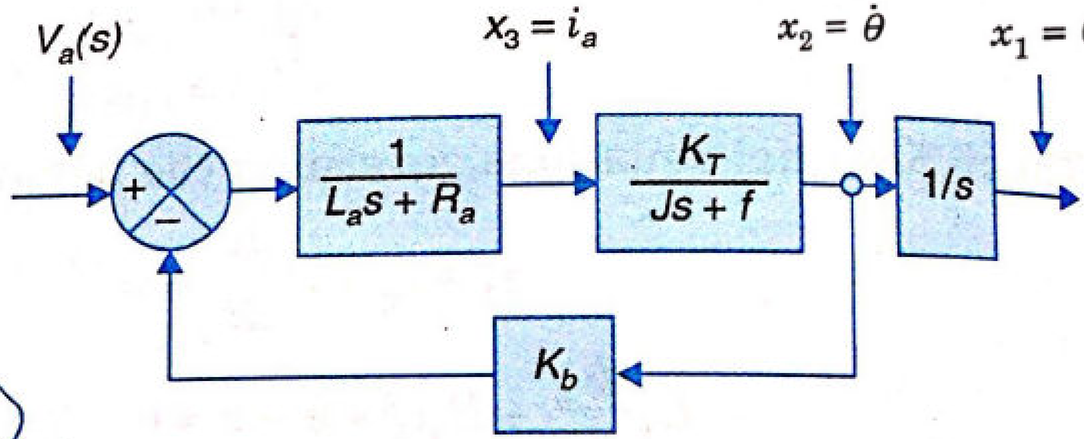


(a) (b)

$$x_1 = \theta\,;\, x_2 = \dot{\theta}\,;\text{ and } x_3 = i_a$$



(a)



(b)

$$\frac{1}{s}, \frac{K_T}{Js + f} \text{ and } \frac{1}{L_a s + R_a}$$

$$\dot{x}_1 = x_2$$

$$J\dot{x}_2 + fx_2 = K_T x_3$$

$$V_a - K_b x_2 = R_a x_3 + L_a \dot{x}_3$$

$$\dot{x}_1 = x_2$$

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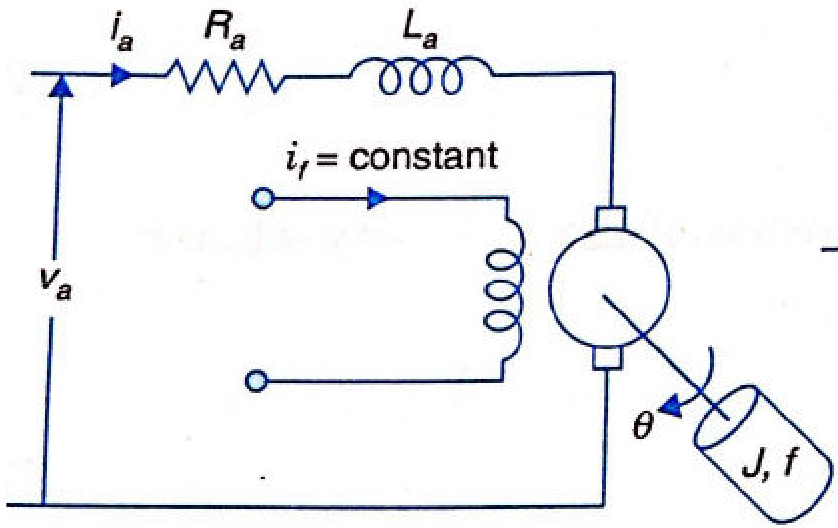
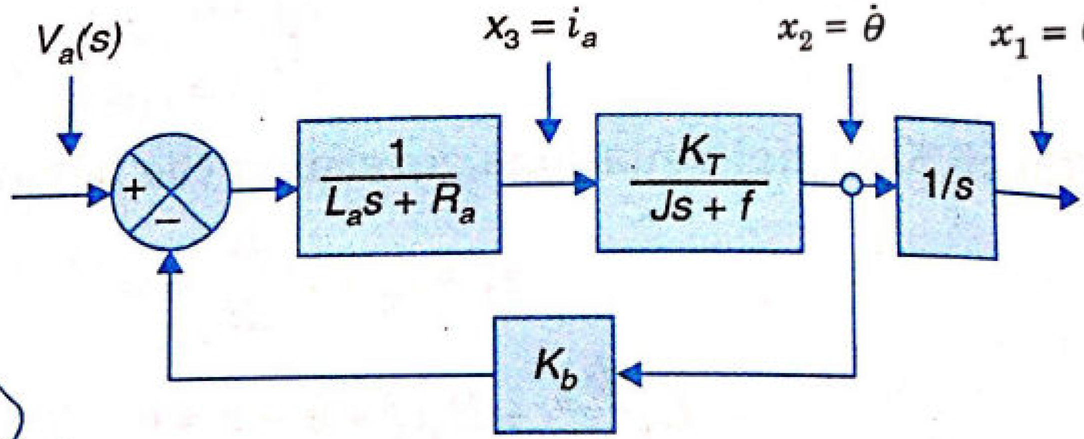


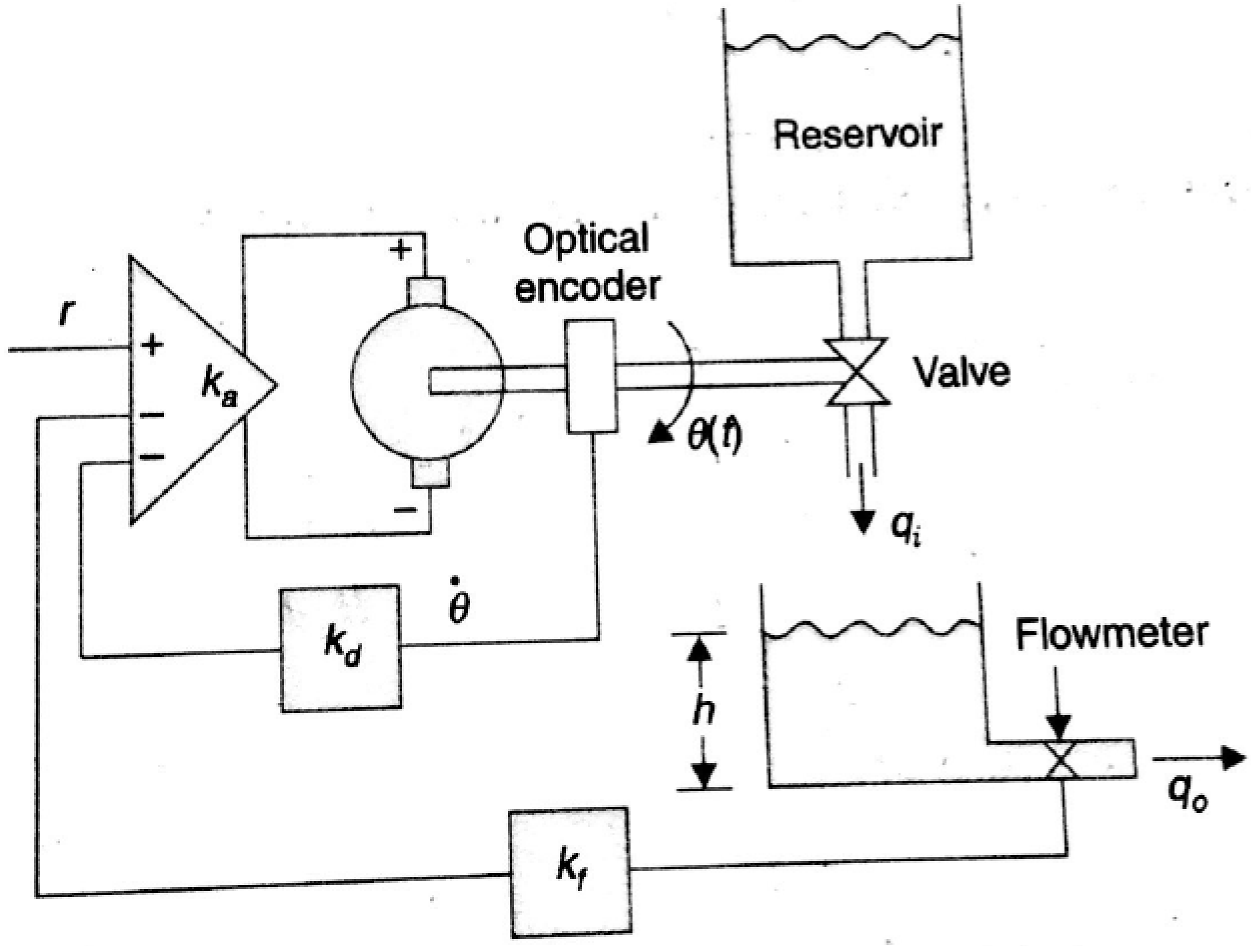
(a)



(b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & K_T/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} v_a \qquad y = \theta = x_1 = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Control Flow

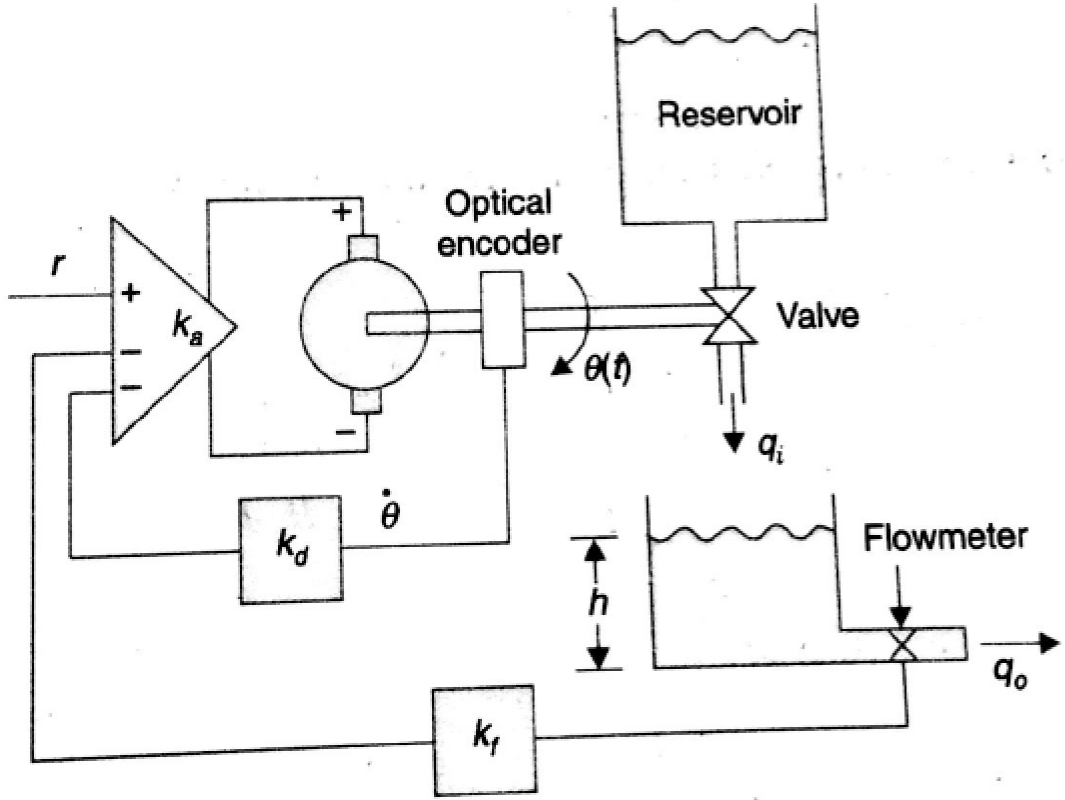


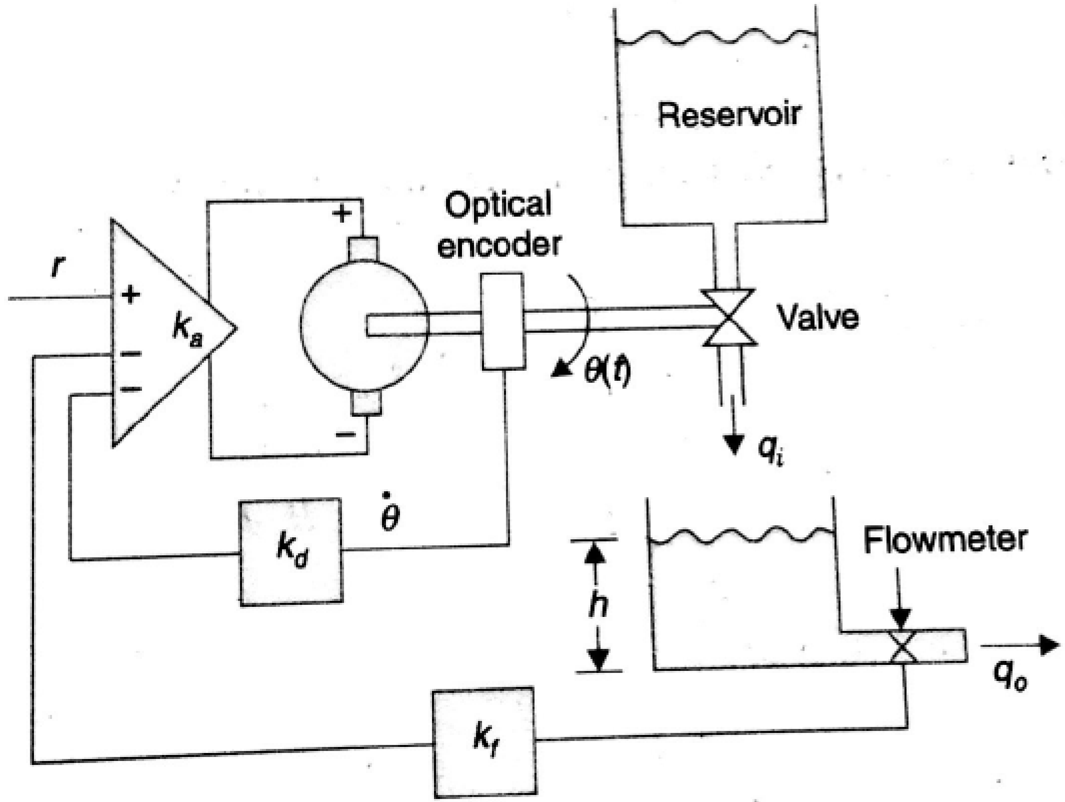
$k_a = 25,\ k_p = 1,\ k_d = 0.005$

$k_m = 5,\ J = 0.05,\ R_a = 1\ \Omega$

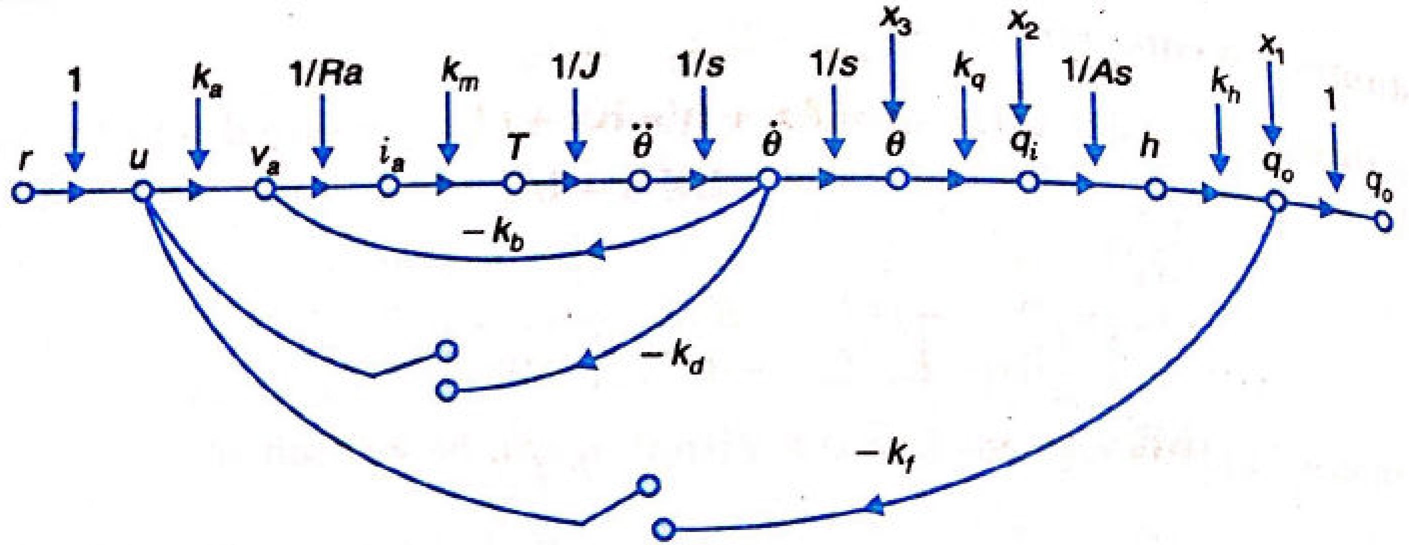
$q_i = k_q\theta,\ k_q = 8$, tank area $A = 50\ \text{m}^2$

$q_0 = k_h h,\ k_h = 225,\ k_f = 0.25.$





$$x_1 = q_0, x_2 = q_i, x_3 = \theta$$



$$\dot{x}_i = \frac{k_h}{A} x_2$$

$$\dot{x}_2 = k_q x_3$$

$$\dot{x}_3 = (rk_a - k_b x_3)\,\frac{k_m}{R_a J}$$

$$= \left(\frac{k_a k_m}{R_a J}\right) r - \left(\frac{k_b k_m}{R_a J}\right) x_3$$

$$y = q_0 = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_h}{A} & 0 \\ 0 & 0 & k_q \\ 0 & 0 & -\left(\frac{k_b k_m}{R_a J}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{vmatrix} 0 \\ 0 \\ \frac{k_a k_m}{R_a J} \end{vmatrix} [u]\,;\; u = r$$

$$y = [1\;0\;0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$