

Control System II

EE 412

Lecture 3

Dr. Mostafa Abdel-geliel

State-Space Representations in Canonical Forms.

1- Controllable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1} \dot{u} + b_n u$$

Special Case

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

Assume

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} \dot{y} + a_n y = bu$$

$$x_1 = y$$

$$x_2 = \dot{y}$$

.....

$$x_n = y^{(n-1)}$$

then



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

.....

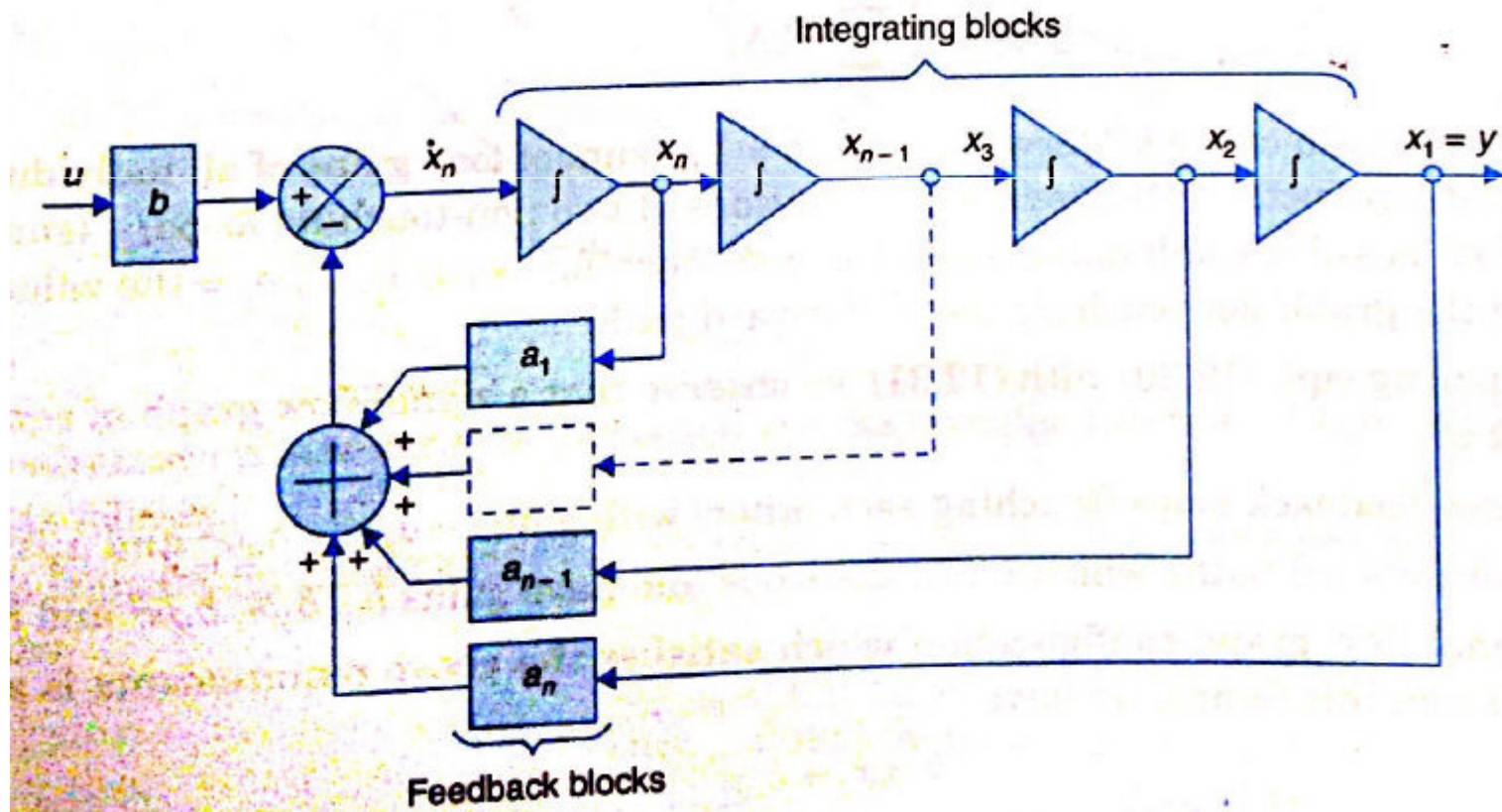
$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + bu$$

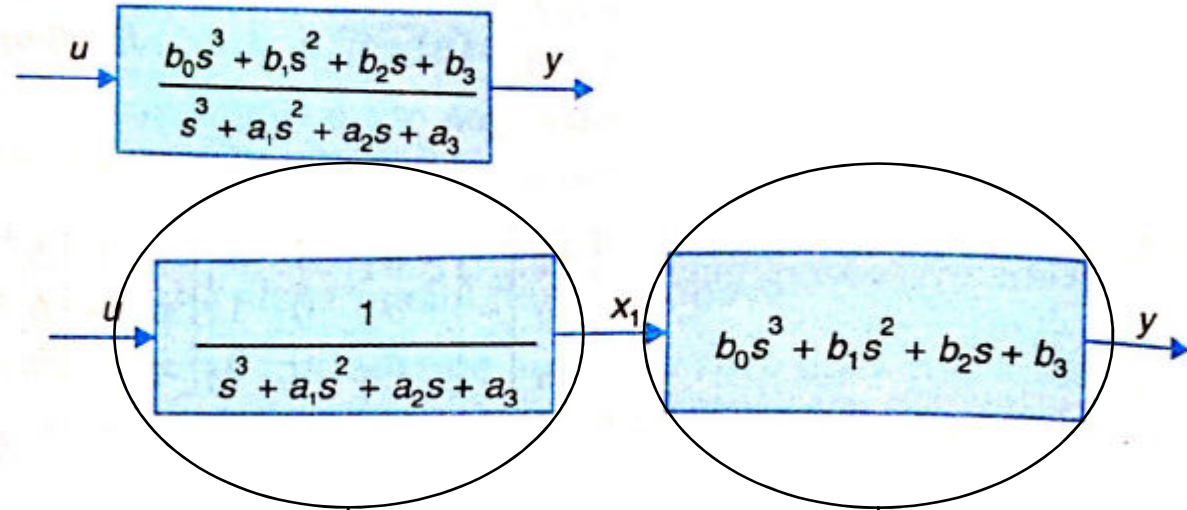
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\dots\dots\dots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + bu \end{aligned}$$



General Case



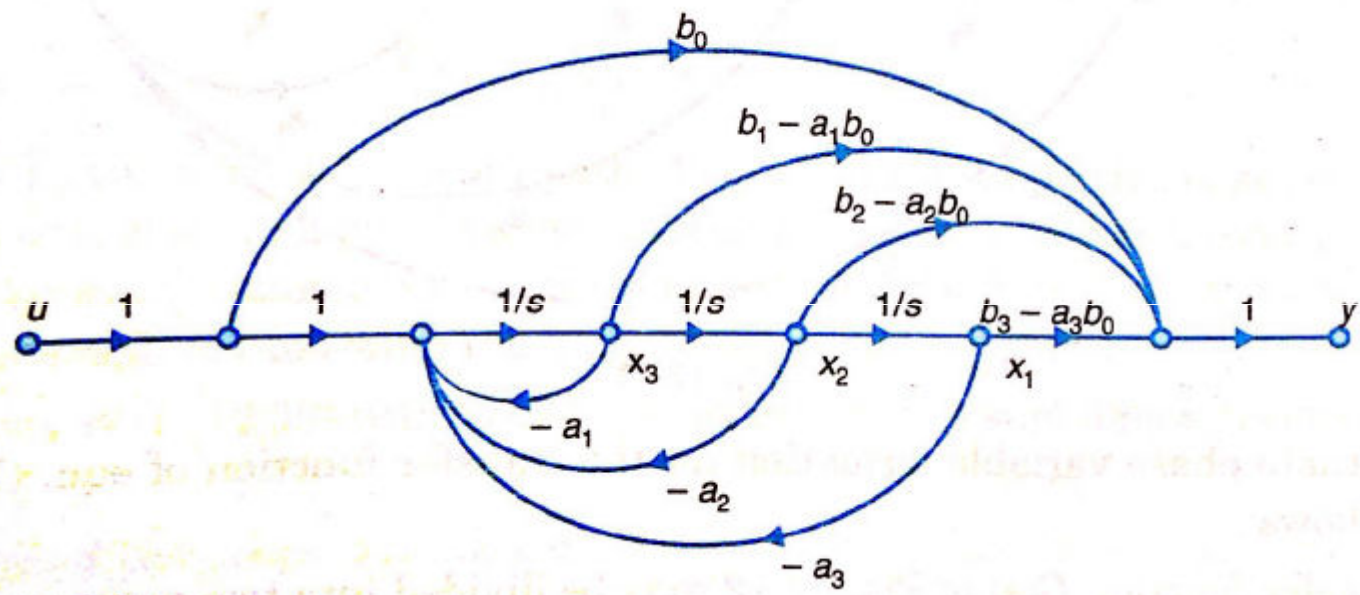
$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + a_1s^2 + a_2s + a_3}$$

$$\frac{Y(s)}{X_1(s)} = b_0s^3 + b_1s^2 + b_2s + b_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\begin{aligned} y &= b_0\ddot{x}_1 + b_1\dot{x}_1 + b_2x_1 + b_3x_1 \\ &= b_0(-a_3x_1 - a_2x_2 - a_1x_3 + u) + b_1x_3 + b_2x_2 + b_3x_1 \\ &= (b_3 - a_3b_0)x_1 + (b_2 - a_2b_0)x_2 + (b_1 - a_1b_0)x_3 + b_0u \end{aligned}$$

$$y = [(b_3 - a_3b_0) \quad (b_2 - a_2b_0) \quad (b_1 - a_1b_0)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0u$$



Controllable Canonical Form General case

$$\frac{Y(s)}{U(s)} = \frac{b_0s^n + b_1s^{n-1} + \dots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_n - a_n b_0 \quad \vdots \quad b_{n-1} - a_{n-1} b_0 \quad \vdots \quad \dots \quad \vdots \quad b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

2- Observable Canonical Form

Prove

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

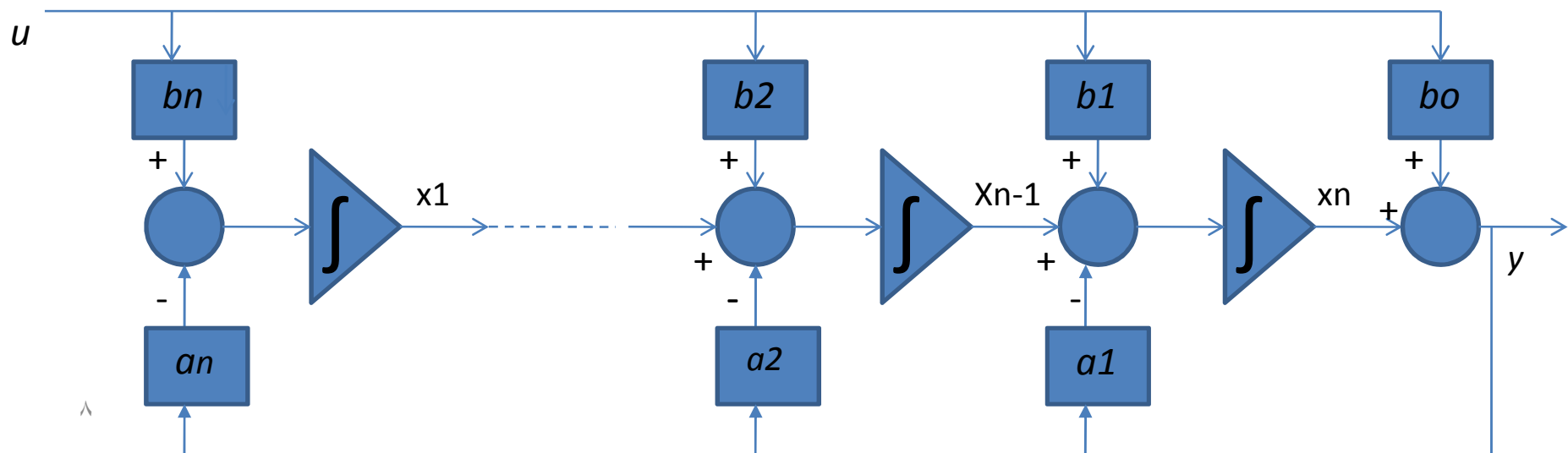
$$\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + a_2 \frac{d^{n-2}}{dt^{n-2}} y + \dots + a_n y = b_0 \frac{d^n}{dt^n} u + b_1 \frac{d^{n-1}}{dt^{n-1}} u + b_2 \frac{d^{n-2}}{dt^{n-2}} u + \dots + b_n u$$

rearrange

$$\frac{d^n}{dt^n} y = b_0 \frac{d^n}{dt^n} u + \frac{d^{n-1}}{dt^{n-1}} (b_1 u - a_1 y) + \frac{d^{n-2}}{dt^{n-2}} (b_2 u - a_2 y) + \dots + (b_n u - a_n y)$$

Integrate both side n times

$$y = b_0 u + \int (b_1 u - a_1 y) dt + \iint (b_2 u - a_2 y) dt + \dots + \iiint_n (b_n u - a_n y) dt$$



General Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad \dots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

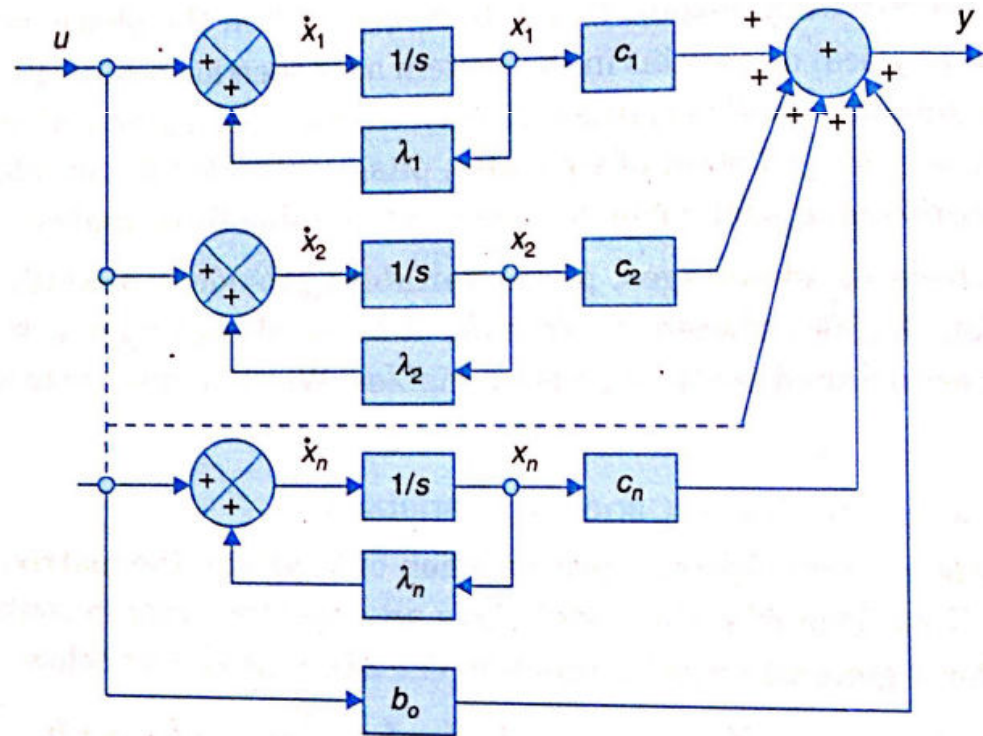
3- Diagonal Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{(s + p_1)(s + p_2) \dots (s + p_n)}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \dots + \frac{c_n}{s + p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad \dots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

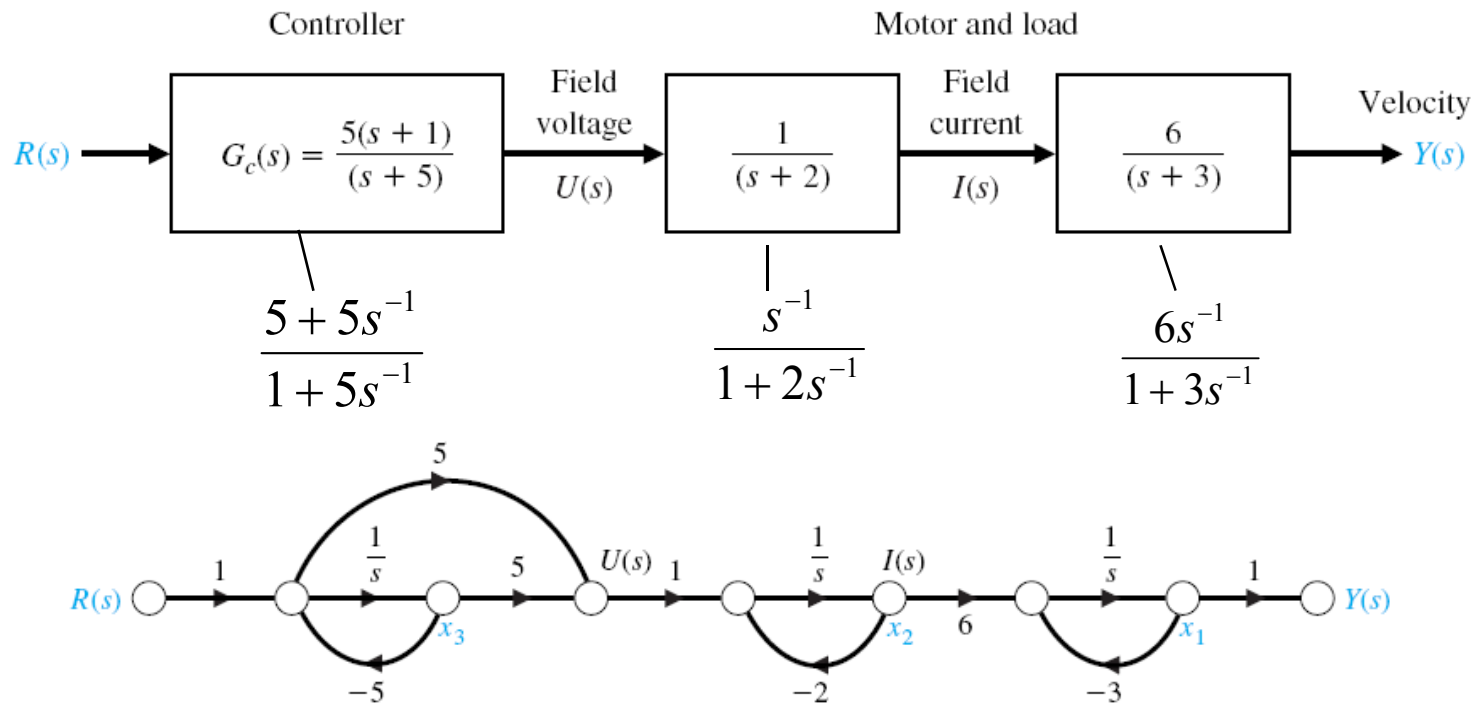


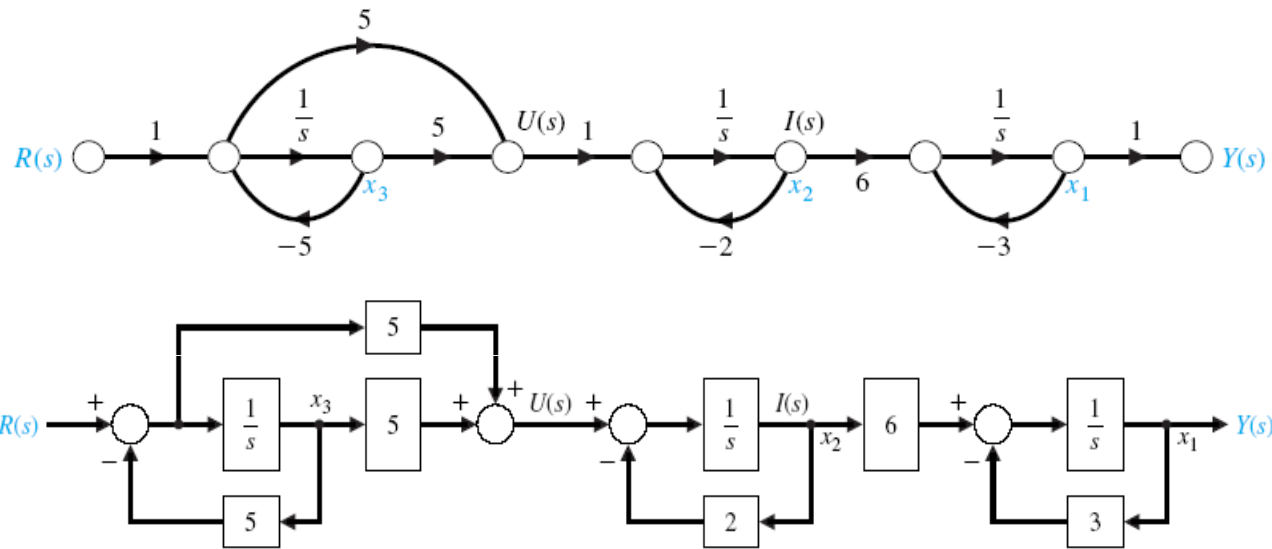
General Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & & 0 \\ & -p_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

4- cascade Form





The corresponding state space equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t)$$

$$y = [1 \quad 0 \quad 0] \mathbf{x}$$

Examples

1- Consider the system given by

$$\frac{Y(s)}{U(s)} = \frac{s + 3}{s^2 + 3s + 2}$$

Obtain state-space representations in the controllable canonical form, observable canonical form, and diagonal canonical form.

Controllable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = [3 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Examples

$$\frac{Y(s)}{U(s)} = \frac{s + 3}{s^2 + 3s + 2}$$

Observable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonal Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [2 \quad -1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Examples

$$\frac{Y(s)}{U(s)} = \frac{10s + 10}{s^3 + 6s^2 + 5s + 10}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [10 \quad 10 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] u$$

Examples

$$\frac{Y(s)}{U(s)} = \frac{25.04s + 5.008}{s^3 + 5.0325s^2 + 25.1026s + 5.008}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.008 & -25.1026 & -5.03247 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [5.24 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Eigenvalues of an $n \times n$ Matrix \mathbf{A} .

The eigenvalues are also called the characteristic roots. Consider, for example, the following matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The eigenvalues of \mathbf{A} are the roots of the characteristic equation, or -1 , -2 , and -3 .

Jordan canonical form

If a system has a multiple poles, the state space representation can be written in a block diagonal form, known as **Jordan canonical form**. For example,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & \vdots & \cdots & \vdots \\ 0 & 0 & -p_1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -p_4 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

Three poles are equal

$$y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$