Control System II
EE 412

Lecture 4

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State-Space and Transfer Function

The SS form

$$\dot{\mathrm{x}} = \mathrm{A}\mathrm{x} + \mathrm{B}\mathrm{u}$$
$$\mathrm{y} = \mathrm{C}\mathrm{x} + \mathrm{D}\mathrm{u}.$$

Can be transformed into transfer function

Tanking the Laplace transform and neglect initial condition then

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + B\mathbf{U}(s) \text{ and } \quad (1)$$

$$\mathbf{Y}(s) = C\,\mathbf{X}(s) + D\mathbf{U}(s) \qquad (2)$$

then

$$s\mathbf{X}(s) - A\mathbf{X}(s) = \mathbf{x}(0) + B\mathbf{U}(s)$$

$$s\mathbf{X}(s) - A\mathbf{X}(s) = \mathbf{x}(0) + B\mathbf{U}(s)$$

by neglecting intial condition then

$$(sI - A)\mathbf{X}(s) = B\mathbf{U}(s)$$

$$\mathbf{X}(s) = (sI - A)^{-1} B\mathbf{U}(s)$$

sub in 2

$$\mathrm{Y(s)} = \mathrm{C}(sI - A)^{-1} B\mathbf{U}(s) + D\mathbf{U}(s)$$

$$\mathbf{Y}(s)/\mathbf{U}(s) = G(s) = \mathrm{C}(sI - A)^{-1} B + D$$

State-Transition Matrix

We can write the solution of the ***homogeneous*** state equation

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ Laplace transform $s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) \quad \Longrightarrow \quad \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

The inverse Laplace transform $\mathbf{x}(t) = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0)$

Note that $(s\mathbf{I} - \mathbf{A})^{-1} = \dfrac{\mathbf{I}}{s} + \dfrac{\mathbf{A}}{s^2} + \dfrac{\mathbf{A}^2}{s^3} + \cdots$

$$\mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots = e^{\mathbf{A}t}$$

Hence, the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1}$

$$\mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots = e^{\mathbf{A}t}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

State-Transition Matrix $\quad \mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$

where $\mathbf{\Phi}(t)$ is an $n \times n$ matrix and is the unique solution of

$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0) = \mathbf{I}$$

Where

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

Note that $\mathbf{\Phi}^{-1}(t) = e^{-\mathbf{A}t} = \mathbf{\Phi}(-t)$

o

If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix A are distinc $\Phi(t)$

will contain the n exponentials

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$$

$$\Phi(t) = e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & & & 0 \\ & e^{\lambda_2 t} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & e^{\lambda_n t} \end{bmatrix}$$

Properties of State-Transition Matrices.

1. $\Phi(0) = e^{\mathbf{A}0} = \mathbf{I}$

2. $\Phi(t) = e^{\mathbf{A}t} = \left(e^{-\mathbf{A}t}\right)^{-1} = \left[\Phi(-t)\right]^{-1}$ or $\Phi^{-1}(t) = \Phi(-t)$

3. $\Phi(t_1 + t_2) = e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$

4. $\left[\Phi(t)\right]^n = \Phi(nt)$

5. $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0) = \Phi(t_1 - t_0)\Phi(t_2 - t_1)$

Obtain the state-transition matri$\Phi(t)$ of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow \Phi(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\mathbf{\Phi}^{-1}(t) = e^{-\mathbf{A}t} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

the ***NON- homogeneous*** state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \Longrightarrow \quad \dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

and premultiplying both sides of this equation by $e^{-\mathbf{A}t}$

$$e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)] = \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Integrating the preceding equation between 0 and t gives

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

or

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

unit-step function

$$u(t) = 1(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\,d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero, or $\mathbf{x}(0) = \mathbf{0}$, then $\mathbf{x}(t)$ can be simplified to

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] u$$

Prove Transfer function of the given ss

$$\frac{Y(s)}{U(s)} = \frac{10s + 10}{s^3 + 6s^2 + 5s + 10}$$

Solution

$$G(s) = C(sI - A)^{-1} B + D$$

Relation of Different SS Representations of the Same System

For a given system G(s) has two different ss representations

$$\text{Rep.1}: \mathrm{M}_1 : \mathbf{x}(t) = A_1\mathbf{x}(t) + B_1\mathbf{u}(t)$$
$$\mathbf{y}(t) = C_1\mathbf{x}(t) + D_1\mathbf{u}(t)$$
$$\text{Rep.2}: \mathrm{M}_2 : \mathbf{x}(t) = A_2\mathbf{z}(t) + B_2\mathbf{u}(t)$$
$$\mathbf{y}(t) = C_2\mathbf{z}(t) + D_2\mathbf{u}(t)$$

Let Z=T x

Where T is the transformation matrix between x and z

For example

take

$$x_1 = y\,, \qquad \mathrm{z}_1 = y$$
$$x_2 = \dot{y}, \qquad \mathrm{z}_2 = \dot{y} + y$$
$$x_3 = \ddot{y}, \qquad \mathrm{z}_1 = \ddot{y} + \dot{y}$$

take

$x_1 = y, \qquad z_1 = y$

$x_2 = \dot{y}, \qquad z_2 = \dot{y} + y$

$x_3 = \ddot{y}, \qquad z_3 = \ddot{y} + \dot{y}$

then

$$z_1 = x_1$$

$$z_2 = x_2 + x_1$$

$$z_3 = x_3 + x_2$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{z} = T\,\mathbf{x}$$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Sub. By z=Tx in rep. 2

$$\dot{z}(t) = T\mathbf{x}(t) = A_2 T\mathbf{x}(t) + B_2\mathbf{u}(t) \text{ mutiply by T}^{-1}$$

$$T^{-1}\dot{z}(t) = T^{-1}T\dot{\mathbf{x}}(t) = T^{-1}A_2 T\mathbf{x}(t) + T^{-1}B_2\mathbf{u}(t)$$

$$y(t) = C_2 T\mathbf{x}(t) + D_2\mathbf{u}(t)$$

Compare with M1;rep.1

$$\text{Rep.1}: \text{M}_1 : \mathbf{x}(t) = A_1\mathbf{x}(t) + B_1\mathbf{u}(t)$$

$$\mathbf{y}(t) = C_1\mathbf{x}(t) + D_1\mathbf{u}(t)$$

then

$$A_1 = T^{-1}A_2 T \qquad A_2 = TA_2 T^{-1}$$

$$B_1 = T^{-1}B_2 \qquad B_2 = TB_1$$

$$C_1 = C_2 T \qquad C_2 = C_1 T^{-1}$$

$$D_1 = D_2 \qquad D_1 = D_2$$