Control System II
EE 412

Lecture 5

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State-Space Diagonalization Function

Eign values and eign vectors

Definition: for a given matrix A, if ther exist a real (complex) λ and a corresponding vector v≠0, such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

Then λ is called eign value and v is the eign vector

i.e.

$$(A - \lambda I)\mathbf{v} = 0$$

And since v≠0

Then

$$(A - \lambda I) = 0$$

i.e

$$\det(A - \lambda I) = 0$$

Eigenvalues of an *n X n* Matrix A.

The eigenvalues are also called the characteristic roots. Consider, for example, the following matrix **A**:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The eigenvalues of **A** are the roots of the characteristic equation, or –1, –2, and –3.

Example

$$A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

then the eign value is the solution of $|\lambda \mathrm{I} - \mathrm{A}| = 0$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -8 & (\lambda + 2) \end{vmatrix} = 0$$

$$\lambda^2 + 2\lambda - 8 = 0 = (\lambda + 4)(\lambda - 2)$$

then

$$\lambda_1 = -4 \quad and \quad \lambda_2 = 2$$

Eign vectors are obtained as

$$at\ \lambda = \text{-}4$$

$$(\lambda_1 \mathrm{I} - \mathrm{A})v_1 = 0$$

$$i.e.$$

$$\begin{bmatrix} -4 & -1 \\ -8 & -2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\therefore \quad v_{12} = -4v_{11}$$

$$let \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$at\ \lambda_2 = 2$$

$$(\lambda_2 \mathrm{I} - \mathrm{A})v_2 = 0$$

$$i.e.$$

$$\begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$\therefore \quad v_{22} = 2v_2$$

$$let \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eign vector matrix

$$V = [\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_n]$$

For all eign values and vectors

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i; \quad i = 0,1,\ldots,n$$

These equations can be written in matrix form

$$A\mathbf{V} = \mathbf{V}\Lambda$$

where

$$\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2 \cdots \mathbf{v}_n]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = diag\{\lambda_i, i = 1,2,\cdots,n\}$$

thus

$$A = \mathbf{V}\Lambda\mathbf{V}^{-1}$$

$$\Lambda = \mathbf{V}^{-1} A\mathbf{V}$$

thus

$$e^{At} = \phi(t) = I + At + A^2 \frac{t^2}{2!} + \ldots$$

$$e^{At} = Ve^{\Lambda t} V^{-1}$$

$$e^{\Lambda t} = \phi(t) = I + \Lambda t + \Lambda^2 \frac{t^2}{2!} + \ldots$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} = diag(e^{-\lambda_i t}, i = 1,2,\ldots,n)$$

Then for a given system has a system matrix A and a state vector **X**

The diagonal system matrix Ad and state **Xd**

$$A_d = \Lambda = T^{-1} AT$$

$$\mathbf{x} = T \mathbf{x}_d ; \ \mathbf{x}_d = T^{-1} \mathbf{x}$$

$$T = V = eign \quad vector \quad matrix$$

$$A_d = V^{-1} AV$$

$$B_d = V^{-1} B_1$$

$$C_d = C_1 T$$

$$D_1 = D_2$$

Example 2: find the transformation into diagonal form and the state transition matrix of example1

$$\Lambda = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$e^{At} = V e^{\Lambda t} V^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}^{-1}$$

$$e^{At} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}$$

Discus how to obtain the transformation matrix between two representation

Diagonal Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Alternative Form of the Condition for Complete State Controllability.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times r$ matrix

If the eigenvectors of **A** are distinct, then it is possible to find a transformation matrix **P** such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \lambda_n \end{bmatrix}$$

Controllability and Observability

- Determine and control the system state from the observation of the output over a finite time interval.
- The concepts of ***controllability and observability*** were introduced by Kalman.
- They play an important role in the design of control systems in state space.
- In fact, the conditions of ***controllability and observability*** may govern the existence of a complete solution to the control system design problem.

The vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

are linearly dependent since

$$\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$$

The vectors

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

are linearly independent since

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}$$

implies that

$$c_1 = c_2 = c_3 = 0$$

Note that:

- if an nxn matrix is nonsingular (that is, the matrix is of rank n or the determinant is nonzero) then n column (or row) vectors are linearly independent.
- If the nxn matrix is singular (that is, the rank of the matrix is less than n or the determinant is zero), then n column (or row) vectors are linearly dependent

To demonstrate this, notice that

$$[\mathbf{x}_1 \vdots \mathbf{x}_2 \vdots \mathbf{x}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \text{singular}$$

$$[\mathbf{y}_1 \vdots \mathbf{y}_2 \vdots \mathbf{y}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \text{nonsingular}$$

CONTROLLABILITY

A system is said to be controllable, if every state variable of the process can be controlled to reach a certain objective in a finite time by some unconstrained control u(t)

Or

A system is said to be controllable at time t_0 if there exist a piecewise unconstrained continuous input "u(t)" (control vector) that will transfer the system from any initial state $\mathbf{x}(t_0)$ to any other state (final state) in a finite interval of time; $t_f - t_0 \geq 0$.

Complete State Controllability of Continuous-Time Systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where $\mathbf{x}$ = state vector (n-vector)

u = control signal (scalar)

$\mathbf{A} = n \times n$ matrix

$\mathbf{B} = n \times 1$ matrix

The system described by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ I to be state controllable at t=t0 if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $0 \leq t \leq t_1$
If every state is controllable, then the system is said to be completely state controllable.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau$$

Applying the definition of complete state controllability just given,

$$\mathbf{x}(t_1) = \mathbf{0} = e^{\mathbf{A}t_1}\mathbf{x}(0) + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}u(\tau)\,d\tau$$

$$\mathbf{x}(0) = -\int_0^{t_1} e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)\,d\tau \quad \Longrightarrow \quad \mathbf{x}(0) = -\sum_{k=0}^{n-1}\mathbf{A}^k\mathbf{B}\beta_k$$

$$\mathbf{x}(0) = -\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{B} \beta_k \quad = -\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right] \begin{bmatrix} \beta_0 \\ \hdashline \beta_1 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline \beta_{n-1} \end{bmatrix}$$

The system is completely state controllable if and only if the vectors

$\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}$ are linearly independent

$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$ is of rank n

Another prove

$$e^{At} = \phi(t) = I + At + A^2\frac{t^2}{2!} + \dots$$

$$x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)Bu(\tau)$$

$$x(t) = e^{At}x(0) + \int_0^t e^{(t-\tau)}Bu(\tau)$$

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-\tau}Bu(\tau)$$

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$$x(t) = (I + At + A^2\frac{t^2}{2!} + \dots)[x(0) + \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

The effect of input u(t)≠0 implies $\int_0^t e^{-A\tau}Bu(\tau)d\tau \neq 0$

$$e^{-A\tau}Bu(\tau) = [I \quad A\tau \quad \frac{A^2\tau^2}{2!} \dots]Bu(\tau)$$

$$[B \quad AB \quad A^2B\cdots]\begin{bmatrix} I & & & \\ & I\tau & & \\ & & I\frac{\tau^2}{2!} & \\ & & & \ddots \end{bmatrix}u(\tau)$$

Then the necessary condition to control x(t)

$$\left| B \quad AB \quad A^2B\cdots A^{n-1}B \right| \neq 0$$

Observability

- Definition:
  - For a dynamic system described by state variable

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}.\end{aligned}$$

  - The state x is said to be observable if for a given input u(t) within a finite time $t_f > t_o$ and the output y(t) and by the knowing of system parameters (A,B,C and D) the initial state x(to) is determined

$$y(t) = C\,x(t) = C(I + At + A^2\frac{t^2}{2!} + \ldots)[x(0) + \int_0^t e^{-A\tau}Bu(\tau)d\tau]$$

$$y(t) - C\int_0^t e^{-A\tau}Bu(\tau)d\tau = C(I + At + A^2\frac{t^2}{2!} + \ldots)x(0)$$

$$y(t) - C\int_0^t e^{-A\tau}Bu(\tau)d\tau = [I\ \ It\ \ I\frac{t^2}{2!}\ldots]\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}x(0)$$

$$x(0) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1}[I\ \ It\ \ I\frac{t^2}{2!}\ldots]^{-1}(y(t) - C\int_0^t e^{-A\tau}Bu(\tau)d\tau)$$

It implies that $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ must be nonsingular

If the system is completely observable, then, given the output y(t) over a time interval x(0) $0 \leq t \leq t_1$ determined from Equation

It can be shown that this requires the rank of the nm*n matrix

$$\begin{bmatrix} \mathbf{C} \\ \hdashline \mathbf{CA} \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline \mathbf{CA}^{n-1} \end{bmatrix}$$

to be n.

The system is completely observable if and only if the n*nm matrix

$$[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

is of rank n or has n linearly independent column vectors. This matrix is called the observability matrix.

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$[\mathbf{CB} \mid \mathbf{CAB}] = [0 \quad 1]$$

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Show that the following system is not completely observable:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [4 \quad 5 \quad 1]$$

$$\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; (\mathbf{A}^*)^2\mathbf{C}^*\right] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

Diagonal representation of SS model and its relation to Observability and controlability and

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \mathbf{x}(t)$$

Discuss the relation between Observability and controlability and the coefficient of B and C matri