

Control System II

EE 412

Lecture 6

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Control System Design in State Space

- This Lecture discusses state-space design based on the pole-placement method
- The pole-placement method is somewhat similar to the root-locus method
- in that we place closed-loop poles at desired locations.
- The basic difference is that in the root-locus design we place only the dominant closed-loop poles at the desired locations,
- while in the pole-placement design we place all closed-loop poles at desired locations.

POLE PLACEMENT

we shall present a design method commonly called the *pole-placement* or *pole-assignment technique*. We assume that all state variables are measurable and are available for feedback.

It will be shown that if the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

Design by Pole Placement

Consider a control system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

\mathbf{x} = state vector (n -vector)

y = output signal (scalar)

u = control signal (scalar)

\mathbf{A} = $n \times n$ constant matrix

\mathbf{B} = $n \times 1$ constant matrix

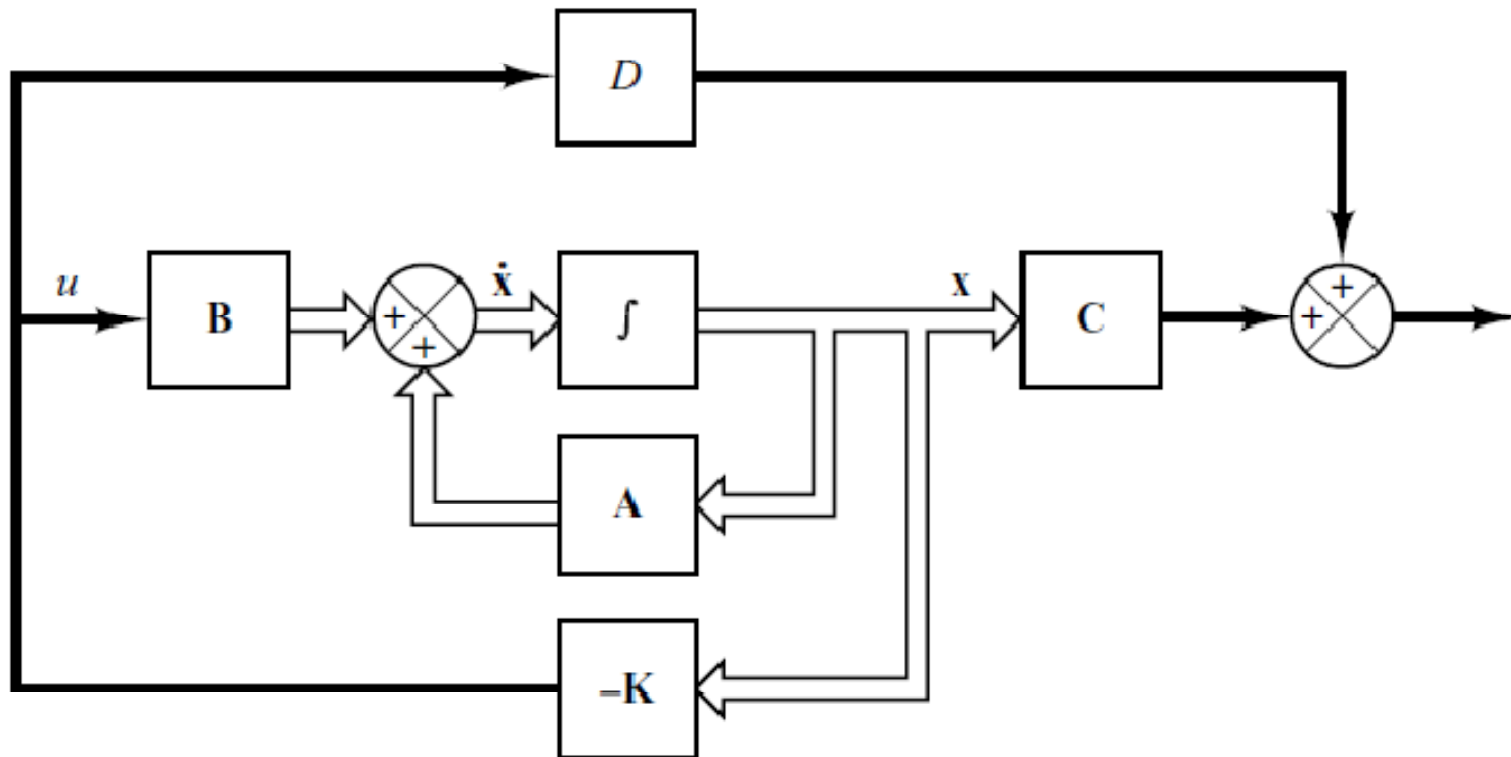
\mathbf{C} = $1 \times n$ constant matrix

D = constant (scalar)

We shall choose the control signal to be

$$u = -\mathbf{K}\mathbf{x}$$

This means that the control signal u is determined by an instantaneous state. Such a scheme is called state feedback. The $1 \times n$ matrix \mathbf{K} is called the state feedback gain matrix. We assume that all state variables are available for feedback. In the following analysis we assume that u is unconstrained.



$$u = -\mathbf{K}\mathbf{x}$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t)$$

The solution of this equation is given by

$$\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{BK})t}\mathbf{x}(0)$$

where $\mathbf{x}(0)$ is the initial state caused by external disturbances. The stability and transient response characteristics are determined by the eigenvalues of matrix $\mathbf{A}-\mathbf{BK}$.

If the eigen values of matrix $A_f = A - BK$ are negative then the system is stable so K is selected to make the new eigen values as required

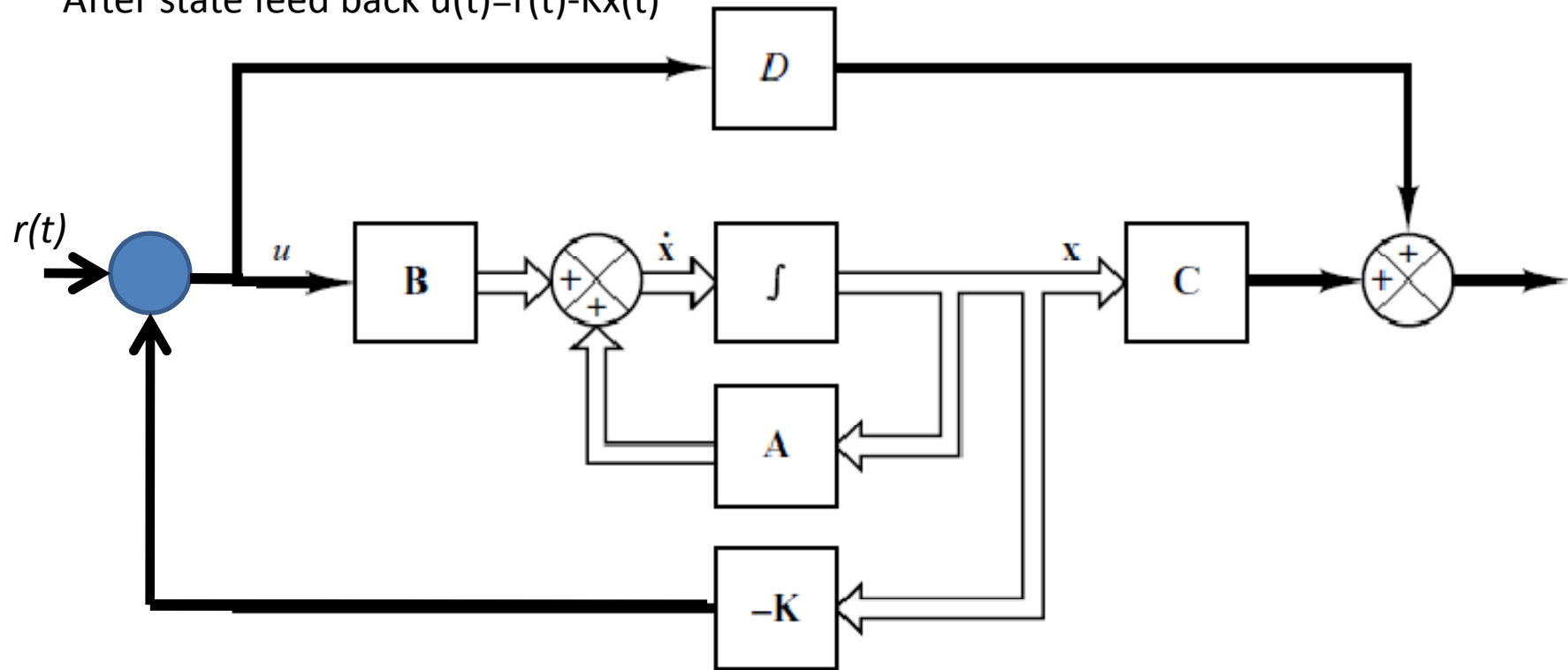
In general

Open loop system

$$\dot{\mathbf{x}}(t) = A \mathbf{x}(t) + B\mathbf{u}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

After state feed back $\mathbf{u}(t) = r(t) - K\mathbf{x}(t)$



$$\dot{\mathbf{x}}(t) = A_f \mathbf{x}(t) + B\mathbf{r}(t)$$

$$\mathbf{y}(t) = C_f \mathbf{x}(t) + D\mathbf{r}(t)$$

$$\mathbf{x}(t) = e^{A_f t} \mathbf{x}(0) + \int_0^t e^{A_f(t-\tau)} B\mathbf{r}(\tau) d\tau$$

$$A_f = A - BK$$

$$C_f = (C - DK)$$

Determination of Matrix K

Using Transformation Matrix T.

Using Ackermann's Formula

Using Direct Substitution Method

Determination of Matrix K Using Direct Substitution Method

If the system is of low order (≤ 3), direct substitution of matrix K into the desired characteristic polynomial may be simpler.

For example, if $n=3$, then write the state feedback gain matrix **K** as

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

Substitute this **K** matrix into the desired characteristic polynomial

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$$

And equate it to

$$(s - \mu_1)(s - \mu_2)(s - \mu_3)$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

Since both sides of this characteristic equation are polynomials in s , by equating the coefficients of the like powers of s on both sides, it is possible to determine the values of k_1 , k_2 , and k_3 .

This approach is convenient if $n=2$ or 3 . (For $n=4, 5, 6, p$, this approach may become very tedious.)

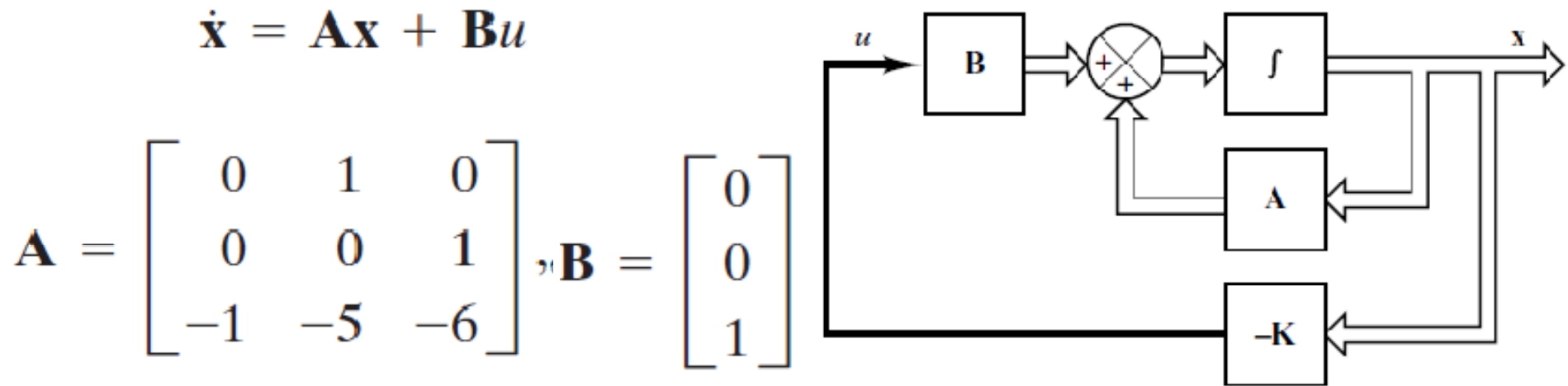
Note that if the system is not completely controllable, matrix K cannot be determined. (No solution exists.)

Choosing the Locations of Desired Closed-Loop Poles

The first step in the pole-placement design approach is to choose the locations of the desired closed-loop poles. The most frequently used approach is to choose such **poles based on experience in the root-locus** design, placing a dominant pair of closed-loop poles and choosing other poles so that they are **far to the left of the dominant closed-loop poles**.

Another approach is based on the quadratic optimal control approach. This approach will determine the desired closed-loop poles such that it balances between the acceptable response and the amount of control energy required.

Consider the regulator system shown in Figure



The system uses the state feedback control $\mathbf{u} = -\mathbf{K}\mathbf{x}$. Let us choose the desired closed-loop poles at

$$s = -2 + j4, \quad s = -2 - j4, \quad s = -10$$

Determine the state feedback gain matrix \mathbf{K} .

First, we need to check the controllability matrix of the system. Since the controllability matrix \mathbf{M} is given by

$$\mathbf{M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we find that $|\mathbf{M}| = -1$, and therefore, $\text{rank } \mathbf{M} = 3$. Thus, the system is completely state controllable and arbitrary pole placement is possible.

By defining the desired state feedback gain matrix \mathbf{K} as

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

and equating $|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$ to the desired characteristic equation, we obtain

$$\begin{aligned} |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| &= \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \right| \\ &= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{vmatrix} \\ &= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1 \\ &= s^3 + 14s^2 + 60s + 200 \end{aligned}$$

$$6 + k_3 = 14, \quad 5 + k_2 = 60, \quad 1 + k_1 = 200$$

$$k_1 = 199, \quad k_2 = 55, \quad k_3 = 8$$

$$\mathbf{K} = [199 \quad 55 \quad 8]$$

Using Transformation Matrix T.

If the SS model of a system $\frac{Y(s)}{U(s)} = \frac{b_0s^n + b_1s^{n-1} + \dots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n}$
 in the form of controllable form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

Discuss the relation between new and old transfer function

Using Ackermann's Formula

$$K = [0 \ 0 \ \dots \ 1] Q_c^{-1} \alpha(A)$$

where

Q_c = controllability matrix

$\alpha(.)$ = new characteristic equation