Linear Control system
MIMO State Feedback Design

Lecture 9

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Topics

- Pole placement in SISO model using state feedback
  - Direct substitution
  - Transformation using Controllable form
  - Ackerman Formula
- Pole placement in case of uncontrollable state
- Pole Placement for MIMO model
  - Difference between SISO and MIMO design
  - Limitation
  - Pole placement using Eign structure technique
- Pole placement using output-feedback

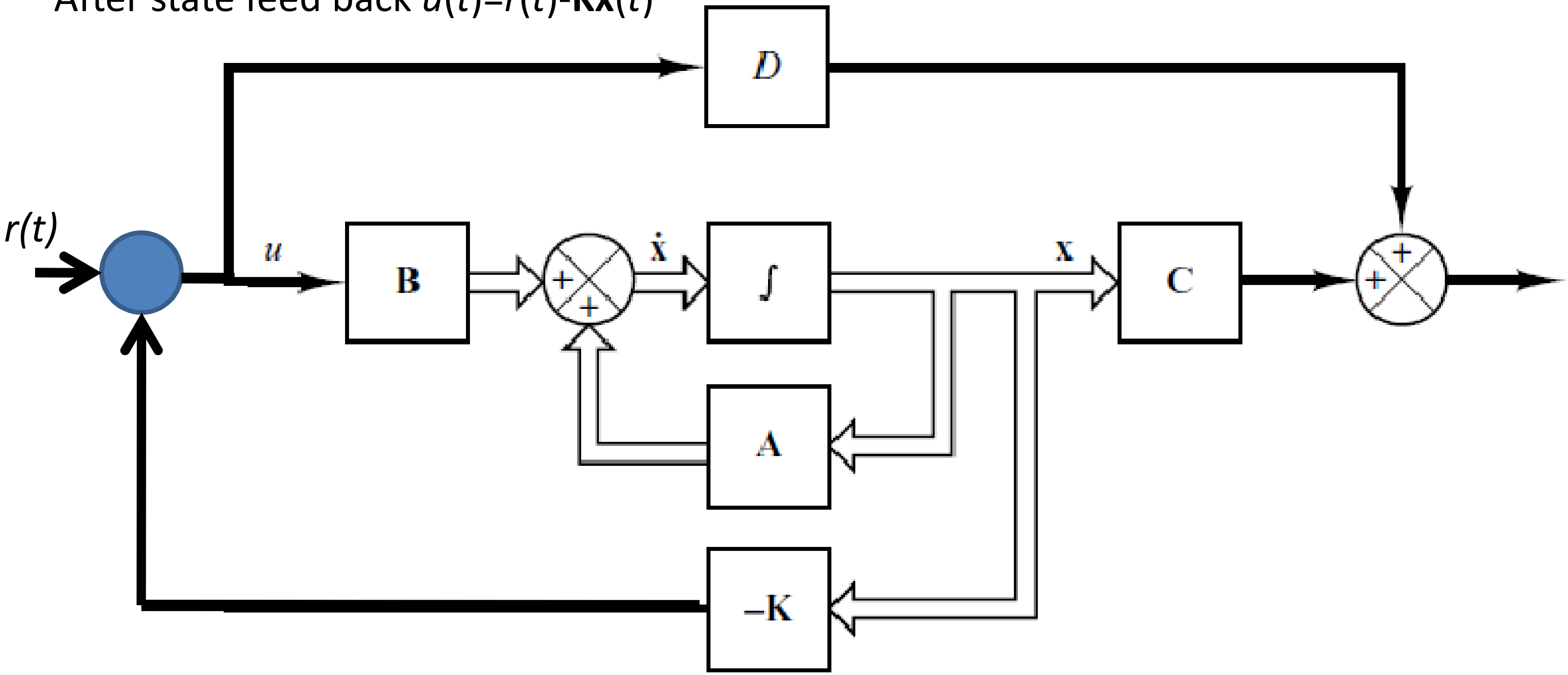
State Feedback

Open loop system

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\mathbf{u}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

After state feed back $u(t)=r(t)-\mathbf{K}\mathbf{x}(t)$



Closed loop system

$$\dot{\mathbf{x}}(t) = A_f\mathbf{x}(t) + B\mathbf{r}(t)$$

$$\mathbf{y}(t) = C_f\mathbf{x}(t) + D\mathbf{r}(t)$$

$$A_f = A - BK$$

$$C_f = (C - DK)$$

<u>The problem</u> here is to Find the feedback vector **K** so that the closed loop eign values (poles) will be place at

$$\lambda_1, \lambda_2, \cdots, \lambda_n$$

The selection of the closed loop eign values is depend on either

1- the system stability i.e. to stabilize the unstable system

2- transient performance; **the dominant closed-loop poles**

3- the quadratic optimal control approach

The Solution $$\mathbf{x}(t) = e^{A_f t}\mathbf{x}(0) + \int_0^t e^{A_f (t-\tau)} B\mathbf{r}(\tau) d\tau$$

Improving the steady state error is improved using either

1- forward gain $u(t)=k_d\, r(t)-\mathbf{k}_f\, \mathbf{x}(t)$

2- integral part (will discussed later)

Determination of Feedback Matrix K

1- Using Direct Substitution Method

2- Using Transformation Matrix T.

3- Using Ackermann's Formula

Determination of Matrix K Using Direct Substitution Method

If the system is of low order ($n \leq 3$), direct substitution of matrix **K** into the desired characteristic polynomial may be simpler.

For example, if n=3, then write the state feedback gain matrix **K** as

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

Substitute this **K** matrix into the desired characteristic polynomial

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$$

And equate it to

$$(s - \mu_1)(s - \mu_2)(s - \mu_3)$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

Since both sides of this characteristic equation are polynomials in s, by equating the coefficients of the like powers of s on both sides, it is possible to determine the values of k1, k2 , and k3 .

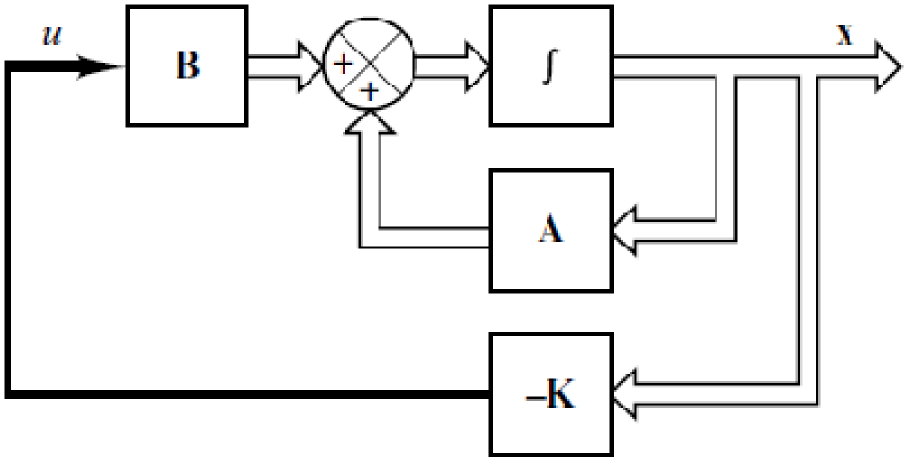
This approach is convenient if n=2 or 3. (For n=4, 5, 6, p , this approach may become very tedious.)

Note that if the system is not completely controllable, matrix K cannot be determined. (No solution exists.)

Consider the regulator system shown in Figure

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



The system uses the state feedback control **u=–Kx.** Let us choose the desired closed-loop poles at

$$s = -2 + j4, \qquad s = -2 - j4, \qquad s = -10$$

Determine the state feedback gain matrix **K.**

First, we need to check the controllability matrix of the system. Since the controllability matrix **M** is given by

$$\mathbf{M} = \begin{bmatrix} \mathbf{B} & \vdots & \mathbf{AB} & \vdots & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we find that |M|=–1, and therefore, rank M=3. Thus, the system is completely state controllable and arbitrary pole placement is possible.

By defining the desired state feedback gain matrix K as

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

and equating $|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$ th the desired characteristic equation, we obtain

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{vmatrix}$$

$$= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

$$= s^3 + 14s^2 + 60s + 200$$

$$6 + k_3 = 14, \qquad 5 + k_2 = 60, \qquad 1 + k_1 = 200$$

$$k_1 = 199, \qquad k_2 = 55, \qquad k_3 = 8$$

$$\mathbf{K} = [199 \quad 55 \quad 8]$$

Using Transformation Matrix T.

If the SS model of a system $$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

in the form of controllable form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

Discuss the relation between new and old transfer function

$$\mathbf{x} = T\mathbf{x}_c$$

$$A_c = T^{-1}AT; \quad B_c = T^{-1}B$$

$$C_c = CT; \quad K = K_c T^{-1}$$

the transformation between any form and controllable cannonical form can be obtained as follow

let $U = [b \quad Ab \quad \cdots \quad A^{n-1}b]$ is the controlability matrix of given system and

$\overline{U} = [\overline{b} \quad \overline{A}\overline{b} \quad \cdots \quad \overline{A}^{n-1}\overline{b}]$ is the controlability matrix of given system in controllable form. then

the charactristic equation in the form

$$s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

$$t_1 = B$$

$$t_2 = At_1 + a_{n-1}t_1 = AB + a_{n-1}B$$

$$t_3 = At_2 + a_{n-2}t_1 = A^2B + a_{n-1}AB + a_{n-2}B$$

$$\vdots$$

$$t_n = At_{n-1} + a_1t_1 = A^{n-1}B + a_{n-1}A^{n-2}B + \cdots + a_2AB + a_1B$$

$$T = [t_n \quad t_{n-1} \quad \cdots \quad t_1]$$

$$\overline{K} = [\alpha_0 - a_0 \quad \alpha_1 - a_1 \quad \cdots \quad \alpha_{n-1} - a_{n-1}]$$

$$K = \overline{K}T^{-1}$$

where

a_i are the cofficient of the given charactristic equation

$$s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

α_i are the cofficient of the required charactristic equation

$$s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0$$

example : design a state feedback for the given system where the new eign values will be placed at

-1, -2 and -3 which means the new char. eq

$$\Delta_c(s) = s^3 + 6s^2 + 11s + 6$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 6 & -3 \\ -1 & -1 & 1 \\ -2 & 2 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

solution

$$\Delta(s) = |sI - A| = s^3 - 3s + 2;$$

$$\lambda_A = \{1, 1, -2\}$$

$$\begin{aligned} \bar{K} &= [\alpha_0 - a_0 \quad \alpha_1 - a_1 \quad \cdots \quad \alpha_{n-1} - a_{n-1}] \\ &= [4 \quad 14 \quad 6] \end{aligned}$$

$$t_1 = B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad t_2 = AB + a_2 B = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix};$$

$$t_3 = A^2 B + a_2 AB + a_1 B = \begin{bmatrix} -5 \\ -6 \\ -13 \end{bmatrix}$$

$$T = \begin{bmatrix} -5 & 4 & 1 \\ -6 & -1 & 1 \\ -13 & 0 & 1 \end{bmatrix}$$

$$K = \overline{K} T^{-1}$$

$$= [4 \quad 14 \quad 6] \begin{bmatrix} -5 & 4 & 1 \\ -6 & -1 & 1 \\ -13 & 0 & 1 \end{bmatrix}^{-1}$$

$$U = \begin{bmatrix} b & Ab & A^2 b \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & -1 & -3 \\ 1 & 0 & -10 \end{bmatrix}$$

$$\overline{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix};$$

Using Ackermann's Formula

$$K = [0 \ \ 0 \ \dots \ 1] \, Q_c^{-1} \, \alpha(A)$$

where

Q_c = controlability matrix

$\alpha(.)$ = new charactristic equation

regarding to the previous example

$$K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} Q_c^{-1} \left[A^3 + 6A^2 + 11A + 6I \right]$$

$$Q_c^{-1} = \begin{bmatrix} 1 & 4 & -2 \\ 1 & -1 & -3 \\ 1 & 0 & -10 \end{bmatrix}^{-1}$$

$$K = [5 \quad 6 \quad -5]$$

Uncontrollable System

The assignment of all eignvalues by using state feedback is possible **<u>if and only if</u>** the system is **controllable**.
If the system has some uncontrollable state, it is possible to find a similarity transformation to decompose the system into controllable and noncontrollable parts. That means A and b matrix will be transformed into

where $\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}, \bar{b} = \begin{bmatrix} \bar{b}_1 \\ 0 \end{bmatrix}, \bar{c} = [\bar{c}_{11} \ \bar{c}_{12}]$

$$\bar{A} = Q^{-1}AQ$$

$$\bar{b} = Q^{-1}b; \bar{c} = cQ$$

$\bar{A}_{11}, \bar{b}_1$ is the controllable

then the eigns of $\bar{A}_{11}$ can be orbitrary assigned

$\bar{A}_{22}$ is the uncontrollable part *hence*

$$\Delta(s) = c(sI - A)^{-1}b = C_{11}(sI - \overline{A}_{11})^{-1}b_1$$

$$Q = [q_1 \cdots q_p \, v_1 \quad \cdots \quad v_{n-p}]$$

is a nonsingular matrix where q_1 to q_p represent the controlable state they are obtained from controlability matrix

$$\mathrm{U} = [\mathrm{b} \quad \mathrm{Ab} \quad \mathrm{A}^2 B \quad \cdots \qquad A^{n-1}b]$$

Select the linear independent columns (q_i) in U with number equal rank(U)

And choose the V_i where it is independent and Q matrix has inversion

Example

$$A=\begin{bmatrix}-5 & -10 & 10\\ 2 & -1 & -2\\ 0 & -4 & 1\end{bmatrix};\quad B=\begin{bmatrix}1 & 4\\ 1 & 0\\ 1 & 2\end{bmatrix}; C=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$$U=\begin{bmatrix}1 & 4 & -5 & 0 & 5 & -20\\ 1 & 0 & -1 & 4 & -3 & 8\\ 1 & 2 & -3 & 2 & 1 & -14\end{bmatrix};\quad rank(U)=2$$

$$Q=[u_1 \quad u_2 \quad v_1];\quad let\ v_1=[1 \quad 0 \quad 0]^T$$

$$Q=\begin{bmatrix}1 & 4 & 1\\ 1 & 0 & 0\\ 1 & 2 & 0\end{bmatrix}, and \quad Q^{-1}=\begin{bmatrix}0 & 1 & 0\\ 0 & -5.5 & 0.5\\ 1 & 1 & -2\end{bmatrix}$$

$$A_t=Q^{-1}AQ=\begin{bmatrix}1 & 4 & 2\\ -1 & -1 & -1\\ 0 & 0 & -3\end{bmatrix},$$

$$B_t=Q^{-1}B=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}, C_t=CQ=\begin{bmatrix}1 & 4 & 1\\ 1 & 0 & 0\end{bmatrix}$$

Design a state feedback so that closed loop pole can be place at -4, -5 and -6

$$A=\begin{bmatrix}-5 & -10 & 10\\ 2 & -1 & -2\\ 0 & -4 & 1\end{bmatrix};\quad B=\begin{bmatrix}4\\0\\2\end{bmatrix}; C=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

$$U=\begin{bmatrix}4 & 0 & -20\\ 0 & 4 & -8\\ 2 & 2 & -14\end{bmatrix};\quad rank(U)=2$$

$$Q=[u_1 \quad u_2 \quad v_1];\quad let\ v_1=[1 \quad 1 \quad 0]^T$$

$$Q=\begin{bmatrix}4 & 0 & 1\\ 0 & 4 & 1\\ 2 & 2 & 0\end{bmatrix}, and$$

$$A_t=Q^{-1}AQ=\begin{bmatrix}0 & -5 & -3\\ 1 & -2 & 1\\ 0 & 0 & -3\end{bmatrix},$$

$$B_t=Q^{-1}B=\begin{bmatrix}1\\0\\0\end{bmatrix}, C_t=CQ=\begin{bmatrix}4 & 0 & 0\end{bmatrix}$$

$$A_{11}=\begin{bmatrix}0 & -5\\ 1 & -2\end{bmatrix},\quad B_1=\begin{bmatrix}1\\0\end{bmatrix}, C_1=\begin{bmatrix}4 & 0\end{bmatrix}$$

$eign(A)=\{-1\pm j2,-3\}$ stable

but it is uncontrollable

it is seen that the 3rd state is uncontrollable

so it can not move

$eign(A_{f11})=\{-4,-5\}$

using Akerman formula for A_{11}, *then*

$K_{t1}=[7 \quad 1]$

$K_t=[7 \quad 1 \quad 0]$

$K=K_tQ^{-1}=\begin{bmatrix}0.75 & -0.75 & 2\end{bmatrix}$

the new eign value

$eign(A_f)=eign(A-BK)=\{-4,-5,-3\}$

MIMO case
In case of MIMO the feedback matrix **K** has unknown
 parameters more than the system poles.
Each input contains *n* feedback gain so
# of gains=n*m; m=number of inputs

It this case the system may be
1- completely controllable
2- incompletely controllable: use controllability decomposition

1- Completely controllable case

There are two ways to design the feedback
1- arbitrary using the same technique of SISO
2- eignstructure Assignment

Eignstructure Assignment

It depends on the eign value and eign vector concept
The system after feedback should satisfy

$$(A - BK)\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

it can be rearanged

$$[\lambda_i I - A \quad B]\begin{bmatrix} \mathbf{v}_i \\ K\mathbf{v}_i \end{bmatrix} = 0$$

$$\mathbf{p_i} = \begin{bmatrix} \mathbf{v}_i \\ K\mathbf{v}_i \end{bmatrix} = \begin{bmatrix} \mathbf{v}_i \\ \mathbf{q}_i \end{bmatrix}$$

then $\mathbf{p_i}$ is in the NULL space of $[\lambda_i I - A \quad B]$

then

$$KV = Q$$

$$K = QV^{-1}$$

Example

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 \\ -5 & 1 & -3 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u};\ y = [0 \quad 0 \quad 1]\mathbf{x}$$

$$system\ eign\ value\ \lambda_A = \{-1, 1 \pm j2\}$$

the required eign values are

$$\lambda_{A_f} = \{-2, -3, -4\}$$

Then

$$T_i = [\lambda_i I - A \quad B] \text{ has three matrix for each eign value}$$

$$T_1 = [-2I - A \quad B]; \mathbf{p}_1 = \alpha_{1,1} n_{1,1} + \alpha_{1,2} n_{1,2}$$

$$T_2 = [-3I - A \quad B]; \mathbf{p}_2 = \alpha_{2,1} n_{2,1} + \alpha_{2,2} n_{2,2}$$

$$T_3 = [-4I - A \quad B]; \mathbf{p}_3 = \alpha_{3,1} n_{3,1} + \alpha_{3,2} n_{3,2}$$

$$P = \begin{bmatrix} V \\ Q \end{bmatrix} =$$

$$K = QV^{-1} = \begin{bmatrix} 5.5 & -1.5 & 2.5 \\ 2.5 & -1.5 & 4.5 \end{bmatrix}$$

This can be done using command **Place** in matlab

Application 1

The Magnetic-tape-Drive system is a MIMO system shown in Fig. There is an independently controllable drive motor on each end of the tape; therefore, it is possible to control the tape position over the read head, x_3, as well as the tension in the tape. The tape is modeled to be a linear spring with small amount of viscous damping. The goal of the control system is to enable commanding the tape to specific position over the read head while maintaining a specified tension in the tape at all times. The desired specifications are that the tape position must be adjusted if the tape head is moved 1mm with 1% settling time of 2.50 sec and overshoot less than 20%. The tape tension, T_e, should be controlled to 2 N with constraint that $0 < T_e < 4$. The current is limited to 1A at each drive motor.

It is required to design a state feedback controller to achieve the required performance

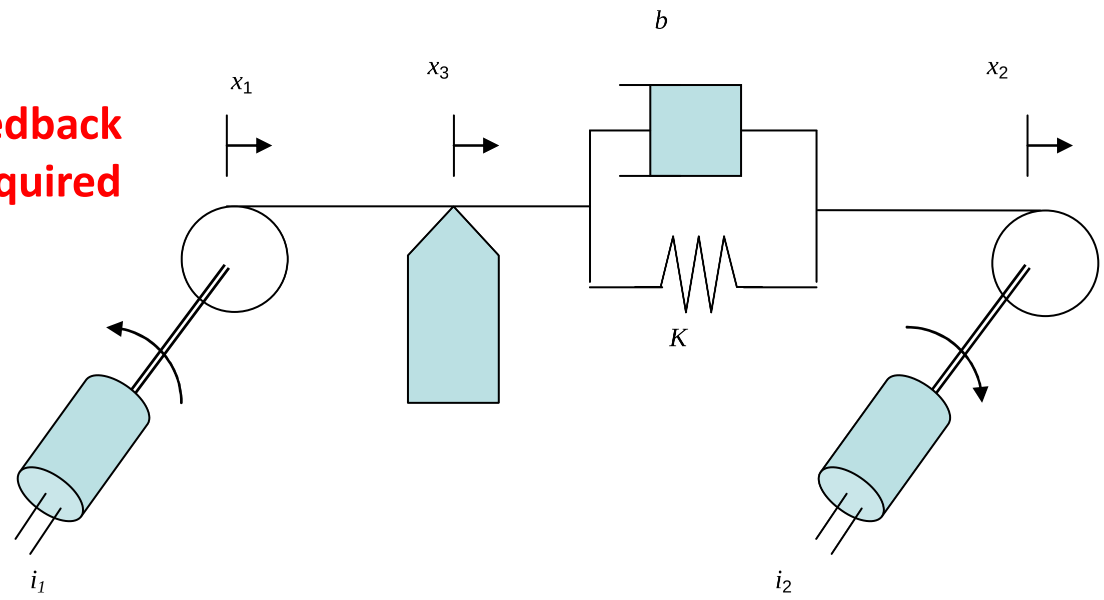


Figure C1: Schematic diagram of magnetic tape drive

The system model is given by

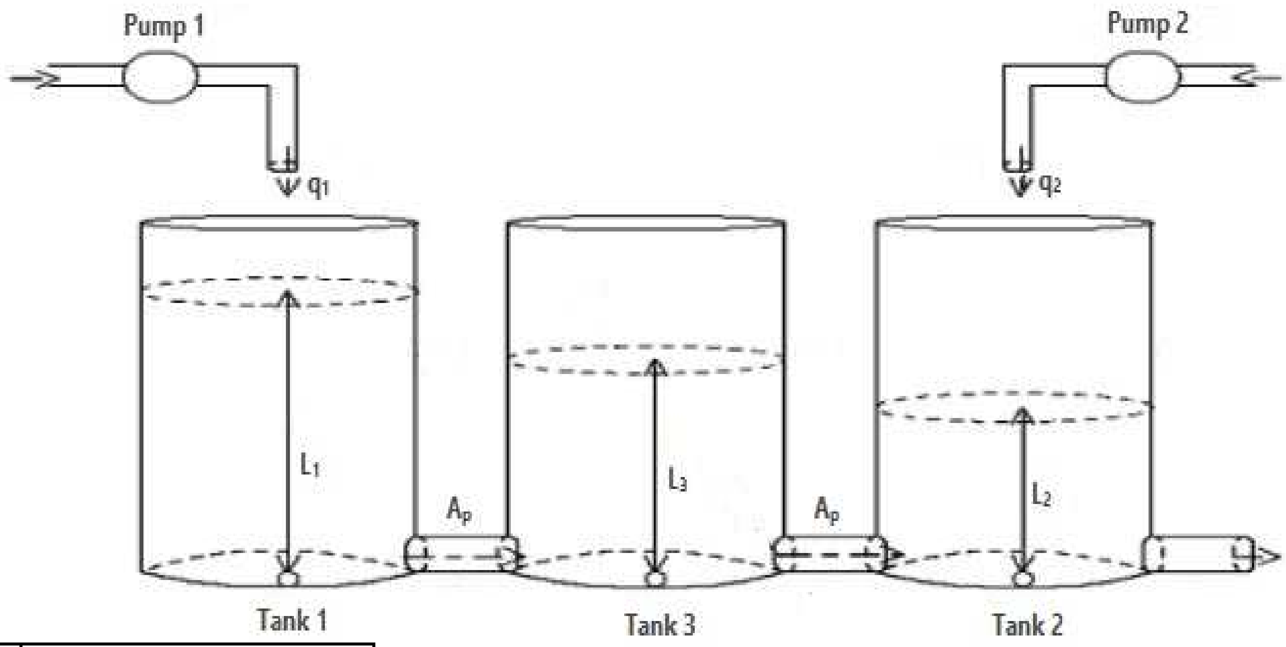
$$J\ddot{\theta}_1 = -T_e r + K_m i_1,$$
$$J\ddot{\theta}_2 = -T_e r + K_m i_2,$$
$$T_e = k(x_2 - x_1) + b(\dot{x}_1 - \dot{x}_2),$$
$$x_3 = (x_2 + x_1)/2,$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 10 \\ 3.315 & -3.315 & -0.5882 & -0.5882 \\ 3.315 & -3.315 & -0.5882 & -0.5882 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 8.533 & 0 \\ 0 & 8.533 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ -2.113 & 2.113 & 0.372 & 0.375 \end{bmatrix}\mathbf{x}$$

where i_1 and i_2 are the current into drive motors 1, 2, respectively,
T_e tension in tape (N),
θ_1 , θ_2 *angular position of motor,* r redias
x_1 *and* x_2 *position of tape over read head (mm),*
J=0.006375 kg.m2, motor and capstan inertia,
r =0.1 m,
K_m*=0.544 N.m/A, motor torque constant,*
k=2.113 N/m, tape spring constant
b=0.375 N sec/m, tape damping constant.
the state vector $\mathbf{x}=[x_1\ x_2\ \omega_1\ \omega_2]^T$*, input vector* $\mathbf{u}=[i_1\ i_2]^T$ *and*
the output vector $\mathbf{y}=[x_3\ T_e]^T$ *is*

Application 2



Tank cross section area (A_t)	0.0154 m^2
Pipe cross section area (A_p)	$5*10^{-5}$ m^2
Outflow coefficient (μ_{mn})	$\mu_{13}=\mu_{32}=0.5$, $\mu_{20}=0.675$
Maximum flow rate (Q_{max})	$1.2*10^{-4}$ m^3/s
Maximum level (L_{max})	0.62 m
Operating point	$Q_1=0.35*10^{-4}$ m^3/s $Q_2=0.375*10^{-4}$ m^3/s $L_1=0.4$ m , $L_2=0.2$ m , $L_3=0.3$ m

$$A_1\frac{dL_1}{dt}=q_1-c_{v13}\sqrt{\delta g(L_1-L_3)}$$

$$A_3\frac{dL_3}{dt}=c_{v13}\sqrt{\delta g(L_1-L_3)}-c_{v32}\sqrt{\delta g(L_3-L_2)}$$

$$A_2\frac{dL_2}{dt}=q_2+c_{v32}\sqrt{\delta g(L_3-L_2)}-c_{v20}\sqrt{\delta gL_2}$$

$$where \quad L_1\geq L_3\geq L_2$$

Deduce the linearized state space model
Then design state feedback that adjust
The input flow q_1 and q_2 in order to make
L1=0.4 m
L2=0.2 m
L3=0.3 m
$q_{1o}= 0.35*10^{-4}$ m^3/s
$q_{20}=0.375*10^{-4}$ m^3/s

1- Output feedback design
2- Observer design