On The Soliton Transmission in Nonlinear Inhomogeneous Graded-Refractive Index Media

FARAG Z. EL-HALAFAWY, EL-SAYED A. EL-BADAWY, MOHAMED A. EL-GAMMAL, AND MOSTAFA H. ALY HASAN

Abstract—**This paper investigates the soliton transmission of optical pulses in nonlinear inhomogeneous graded-refractive index fibers (biquadratic profile). Avoiding the linear fashion approximation, the radial dependence of the field is treated. The obtained solutions are parametrically controlled in such a manner that both the light and dark solitons exist in both normal and anomalous dispersion regions. The controlling parameters affect the power required to achieve soliton propagation, the bit rate, and the group velocity.**

I. Introduction

Recently, two decades ago, the propagation of solitary waves in nonlinear dispersive optical fibers has become of great importance. These pulse-type stationary solutions of nonlinear dispersive wave equations has attracted the attention both mathematically and physically [1]-[3].

Solitons in optical communications are now considered to have a potential application to high-bit-rate transmission systems [4] with an additional important merit of automatic pulse reshaping during the course of transmission [5]-[8]. The radial dependence of the field has been linearly treated in the previous theories [9], [10] where the equation for the scalar envelope $\varphi(z, t)$ of the field that depends on the axial position z and the time t has the form

$$a \frac{d^2\varphi}{d\eta^2} + b\varphi + c\varphi^3 = 0 \tag{1}$$

where $\eta = (z - v_g t)/v_g T$; v_g is the group velocity, T is the pulse duration, and a, b, and c are appropriate constants. This equation yields both light (a is positive) and dark (a is negative) solitons of the form sech $\rho\eta$ and tanh ρ_η, respectively, for anomalous and normal dispersion regions of the fiber. Fibers with either quadratic or step-refractive index in the radial direction had been considered [9], [10].

El-Halafawy *et al.* [11] analyzed the soliton propagation on the basis of the same linear fashion through graded fibers of biquadratic-refractive index $n(r, \omega)$ of the form

$$n(r, \omega) = n(\omega)[1 - \alpha r^2 + m\alpha r^4] \tag{2}$$

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F. Z. El-Halafawy is with the Faculty of Electronic Engineering, University of Menoufia, Menouf 23951 Egypt.
El-S. A. El-Badawy, M. A. El-Gammal, and M. H. Aly Hasan are with the Faculty of Engineering, University of Alexandria, Alexandria, Egypt.
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where ω is the angular frequency, and α and m are positive controlling parameters. The parameter α has a threshold value α_{th} related to fiber radius R. Under this critical value, soliton propagation is impossible. The controlling parameters α and m yield both light and dark solitons for both anomalous and normal dispersion regions of the fibers. The radial dependence was treated exactly for lossless dispersive step-index fiber [12], [13]. Both light and dark solitons are found in the anomalous dispersion region of the fiber. In the present paper we analyze the soliton in dispersive biquadratic-refractive index fiber avoiding the linear fashion treatment of the radial dependence. Section II describes the basic model and deals with the mathematical formulation, analysis, and computational techniques. The results and discussion are exposed in Section III. The main conclusions are pointed out in Section IV.

II. Basic Model and Analysis

It is now a well-established fact that the space–time evolution of the complex amplitude $E(r, Z, t)$ of an optical-field envelope in a glass fiber obeys the nonlinear differential equation [9], [14]:

$$\left[\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} + 2jq\frac{\partial}{\partial z} - q^2 + \left(n^2(r, \omega)/n^2(\omega)\right) \cdot \left\{\left(k_0^2 + 2jk_0k_0' \times \frac{\partial}{\partial t} - (k_0'^2 + k_0k_0'')\frac{\partial^2}{\partial t}\right)\right\}\right] A(r, z, t) = \frac{-2n_2}{n_0} k_0^2 |A|^2 A \tag{3}$$

where

$$E(r, z, t) = A(r, z, t)\, e^{-j\omega t}. \tag{4}$$

The electric field amplitude is given by $A(r, z, t)$. The quantities k_0' and k_0'' are given by

$$k_0' = \left.\frac{\partial k}{\partial \omega}\right|_{\omega = \omega_0} \tag{5}$$

$$k_0'' = \left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega = \omega_0} \tag{6}$$

where

$$k = \omega n(r, \omega)/c. \tag{7}$$

In the present analysis, the index of refraction is assumed under the form

$$n(r, \omega) = n(\omega)(1 - \alpha\rho^2 + \alpha m\rho^4) \tag{8}$$

where $\rho(=r/R)$ is the normalized radial position and R is the fiber radius. The scalar amplitude (envelope) $A(r, z, t)$ is obtained as

$$\left[\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} - \beta^2 + \frac{n^2(r,\omega)}{n^2(\omega)} k_0^2 - (k_0'^2 + k_0 k_0'')\frac{\partial^2}{\partial t^2}\right] \cdot A(r, z, t) = -\frac{2n}{n_0} k_0^2 |A|^2 A. \tag{9}$$

The use of

$$\eta = (z - v_g t)/v_g T \tag{10}$$

and

$$\varphi(r, z, t) = [2n_2 R^2 k_0^2 / n_0]^{1/2} A \tag{11}$$

in (9) yields

$$\left[\nabla_\rho^2 + a\frac{2\partial^2}{\partial\eta^2} + b(1 - \alpha\rho^2 + m\alpha\rho^4)^2 - c\right]\varphi + \varphi^3 = 0 \tag{12}$$

with

$$\nabla_\rho^2 \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}$$

$$a = R^2\{v_g^{-2} - (k_0'^2 + k_0 k_0'')\} T^{-2} \tag{13}$$

$$b = R^2 k_0^2 \tag{14}$$

and

$$c = R^2\beta^2. \tag{15}$$

To obtain either the light soliton solution, φ_L, or the dark one, φ_D, even in and on the same spirit of [13], [14] we represent $\varphi(\rho, \eta)$ under the form

$$\varphi_L(\rho, \eta) = \operatorname{sech}\eta \sum_{n=0}^{n=\infty} R_n(\rho)\operatorname{sech}^{2n}\eta \tag{16}$$

$$\varphi_D(\rho, \eta) = \tanh\eta \sum_{n=0}^{n=\infty} R_n(\rho)\operatorname{sech}^{2n}\eta. \tag{17}$$

These solutions, whatever the value of n, must yield

$$\varphi(1, \eta) = 0, \qquad \text{at } r = R.$$

Substituting (16) in (12) and then equating to zero the coefficient of each power of $\operatorname{sech}^2\eta$ result in the following:

$$\sum_{n=0}^{\infty} S^{2n}\left[R_n + \frac{1}{\rho}R_n + \gamma_n^2 R_n\right] - 2S^2\sum_{n=0}^{\infty}(n+1)(2n+1) \cdot S^{2n}R_0 + S^2\{\Sigma R_n S^{2n}\}^3 = 0 \tag{18}$$

where $S = \operatorname{sech}\eta$ and $\gamma_\eta^2 = b(1 - \alpha\rho^2 + m\alpha\rho^4)^2 - c + a(2n+1)^2$

$$R_0'' + \frac{1}{\rho}R_0 + \gamma_0^2 R_0 = 0 \tag{19}$$

$$R_1'' + \frac{1}{\rho}R_1 + \gamma_1^2 R_1 = 2R_0\left(a + \frac{1}{2}R_0^2\right) \tag{20}$$

and

$$R_2'' + \frac{1}{\rho}R_2 + \gamma_2^2 R_2 = 6R_1\left(2a - \frac{1}{2}R_0^2\right). \tag{21}$$

With the boundary conditions

$$\frac{dR_n}{d\rho} = 0 \text{ at } \rho = 0 \tag{22a}$$

$$R_n = 0 \text{ at } \rho = 1. \tag{22b}$$

Similar relationships can be obtained for dark solitons by substituting (17) in (12).

The linear imaginary part of (3) has been employed to derive an expression for the constant envelope velocity v_g under a radially averaged form:

$$v_g = \frac{\beta}{k_0 k_0'\left(1 + \frac{m^2\alpha^2}{5} - \frac{\alpha^2 m}{2} + \frac{\alpha^2}{3} - \frac{2\alpha m}{3} - \alpha\right)} \tag{23}$$

where β is the phase constant. The use of the perturbation theory to the third order for single-mode fibers yields [15]

$$\beta^2 = k_0^2 - \frac{2k_0}{R}\sqrt{2\alpha} + \frac{2m}{R^2} + \frac{9}{2}\frac{m^2}{\sqrt{2\alpha}R^3k_0^2} + \frac{79m^3}{8\alpha R^2 k_0^2}. \tag{24}$$

For R_n we assume a solution of the form

$$R_n = \sum_{i=0}^{\infty} a_{ni}\rho^i. \tag{25}$$

The use of (25) in (19), (20), and (21) yields, for odd values of subscript i

$$a_{n1} = a_{n3} = a_{n5} = \cdots = 0 \tag{26}$$

for $n = 0$ and even values of i

$$\sum_{j=0} 4(j+1)^2 a_{02(j+1)}\rho^{2j} + \{b(1 - \rho^2 + m\alpha\rho^4)^2 - c + a\} \times \sum_{j=0} a_{02j}\rho^{2j} = 0. \tag{27}$$

Equating the coefficients of the same power of ρ^2 on both sides, one can get

a_{0m} functions of $\{a, b, c, \alpha, m, a_{00}\}$

a_{1m} functions of $\{a, b, c, \alpha, m, a_{00}, a_{10}\}$

a_2m functions of $\{a, b, c, \alpha, m, a_{00}, a_{10}, a_{20}\}$

$\cdots$ etc.

where $m = 2, 4, 6, \cdots$.

The use of the boundary conditions given by (22), with the restriction that $dR/d\rho = 0$, yields the coefficients a_{nm}.

The peak power P_0 of the pulse is obtained as [12]

$$P_0 = \frac{1}{2} n_0 \left(\frac{\epsilon_0}{\mu_0} \right)^{1/2} S A_{\max}^2 \tag{28}$$

where n_0 is the refractive index, S is the cross-sectional area of the fiber, ϵ_0 and μ_0 are, respectively, the permittivity and permeability of free space, and $A_{\max}$ is the maximum electric field which is obtained as follows:

$$\varphi(\rho, \eta) = R_0(\rho) \operatorname{sech} \eta + R_1(\rho) \operatorname{sech} \eta^3 + R_2(\rho) \operatorname{sech}^5 \eta. \tag{29}$$

As $\rho = \eta = 0$, φ possesses a maximum value $\varphi_{\max}$ where

$$\varphi_{\max} = \sum_{n=0} a_{n0}. \tag{30}$$

The use of (30) in (11) yields

$$A_{\max} = \sum_{n=0} a_{n0} / [2n_2 R^2 k_0^2 / n_0]^{1/2}. \tag{31}$$

III. Results and Discussion

The most important expansion coefficient of the functions $R_n(\rho)$ is the coefficient a_{00}, on which all other coefficients of different modes are directly dependent. This coefficient is derived as

$$a_{00} = \sqrt{a} \tag{32}$$

where a is given by (13). The quantity "a" must be positive to give a real solution. Thus a criterion for the existence of a solution can be stated as, "A soliton solution is obtained if the quantity a_{00}^2 is positive." The use of (13) into (32) yields

$$a_{00} = 2R^2 \{ v_g^{-2} - (k_0'^2 + k_0 k_0'') \} / T^2 \tag{33}$$

where v_g, k_0, k_0', and k_0'' are functions of R, α, m, and λ. Thus for fixed R and λ, the parameters α and m control the sign of the quantity α_{00}^2 irrespective of the type of dispersion (normal or anomalous) or the type of soliton (light or dark).

The variations of the peak power P_0 and the group velocity v_g are investigated at $\lambda = 10.6$ μm for different values of the set of parameters α, m, and R. The expansion coefficients of the first three modes are given in Table I.

From this table, it is clear that all the expansions of the different modes are convergent, but it is clearly portrayed that the zero mode ($n = 0$) is the most dominant one. The average value of the coefficients $\langle a_{nm} \rangle$ points to the same conclusion. The variations of the quantity a_{00}^2 against the parameter α at the assumed set of parameters are displayed in Fig. 1, where it is clear that there is a threshold value for the parameter α after which one obtains a stable soliton solution ($a_{00}^2 \geq 0$). Both the variations of the group velocity v_g and the peak power P_0 possess, respectively, a minimum and a maximum with the variations of the fiber radius R and the assumed set of parameters around $R \approx 7.5$ μm as displayed in Figs. 2 and 3.

TABLE I
THE EXPANSION COEFFICIENTS OF THE FIRST THREE MODES WHERE $\lambda = 10.6$ μm, $R = 5\mu$m, $T = 1$ ns, $\alpha = 0.1$, AND $m = 0.55$

$a_{00}=1.428 \times 10^{+1}$	$a_{10}=-1.319 \times 10^{-4}$	$a_{20}=1.127 \times 10^{-7}$
$a_{02}= 1.376$	$a_{12}=1.288 \times 10^{-5}$	$a_{22}=-1.375 \times 10^{-7}$
$a_{04}=1.486$	$a_{14}=2.056 \times 10^{-4}$	$a_{24}=1.191 \times 10^{-8}$
$a_{06}=4.656 \times 10^{-1}$	$a_{16}=-1.174 \times 10^{-4}$	$a_{26}=1.575 \times 10^{-8}$
$a_{08}= 8.164 \times 10^{-2}$	$a_{18}=4.148 \times 10^{-5}$	$a_{28}=-3.032 \times 10^{-9}$
$a_{010}=2.716 \times 10^{-2}$	$a_{10}=-1.599 \times 10^{-5}$	$a_{210}=-6.32 \times 10^{-10}$
$a_{012}=5.31 \times 10^{-3}$	$a_{112}=5.420 \times 10^{-6}$	$a_{212}=7.720 \times 10^{-10}$
$\langle a_{om} \rangle = 1.398$	$\langle a_{1m} \rangle = -7.992 \times 10^{-5}$	$\langle a_{2m} \rangle = 5.128 \times 10^{-8}$

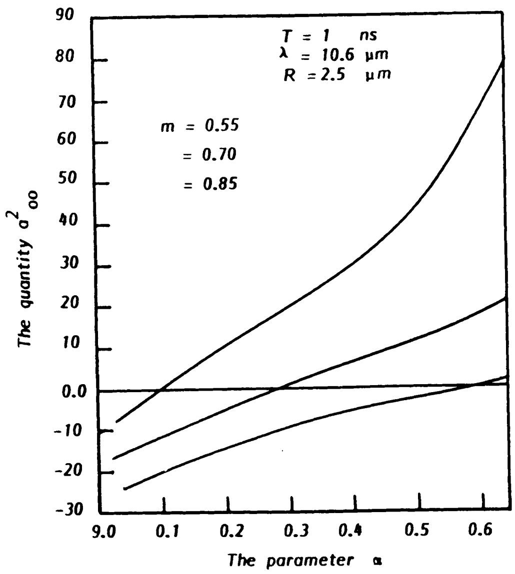

Fig. 1. The variations of the quantity a_{00}^2 with the parameter α.

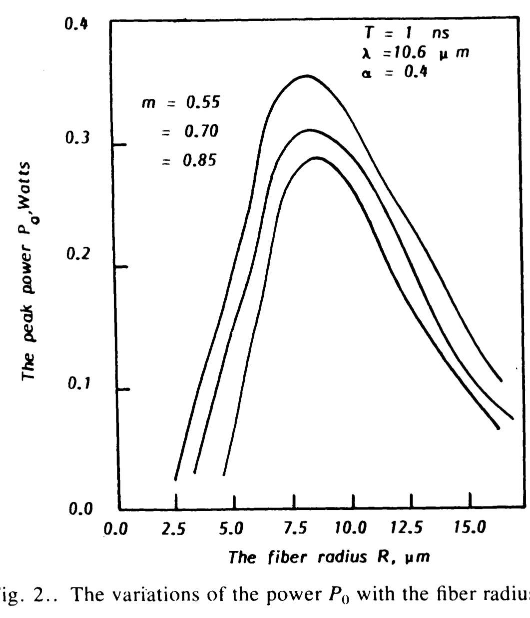

Fig. 2.. The variations of the power P_0 with the fiber radius R.

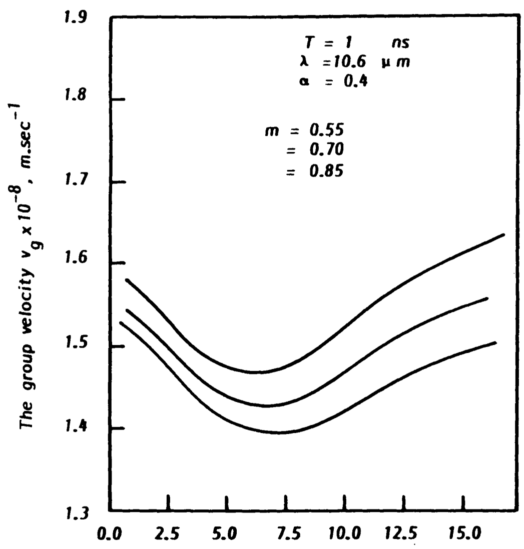

Fig. 3. The variations of the group velocity v_g with the fiber radius R.

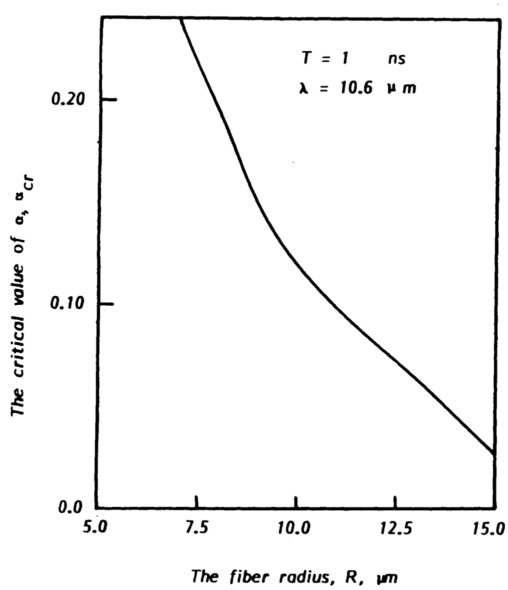

Fig. 4. The variations of the parameters α_{cr} with the fiber radius R.

From the previous results it is noted that, for different radii, there are critical values α_{cr} at which the values of the group velocity and the peak power are independent of the parameter m. The variation of α_{cr} with the fiber radius is displayed in Fig. 4.

IV. Conclusion

For the best design of a single-mode fiber for digital transmission (high v_g and low P_0) at $\lambda = 10.6$ μm, the following requirements are to be fulfilled.

a) The radius R must be smaller than 7.5 μm.

b) The controlling parameters α and m must be tailored with minimum and maximum values, respectively.

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