Arab Academy for Science, Technology and Maritime Transport
College of Engineering and Technology
Department of Basic and Applied Science

Mathematics (2) BA124

Lecture No.13

- Eigenvalues and Eigenvectors
- Cayley-Hamilton Theorem

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Eigenvalues, Eigenvectors and Cayley-Hamilton Theorem

Definition:

Consider the equation $A_{n\times n}X_{n\times 1}=\lambda X$.
which can be written as

$$(A-\lambda I)X=0$$

A *is a square matrix*

λ is a scalar and is called the matrix Eigen value.

X is called the eigen vector . It's a column matrix.

Theorem

The Eigen value λ can be obtained from the characteristic equation $|A-\lambda I|=0$

Sheet 11

No.1-a

$$A=\begin{pmatrix}-1 & 3\\ -2 & 4\end{pmatrix}$$

Solution:

$$A-\lambda I=\begin{pmatrix}-1 & 3\\ -2 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-1 & 3\\ -2 & 4\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}-1-\lambda & 3\\ -2 & 4-\lambda\end{pmatrix}$$

The characteristic equation:

$$|A-\lambda I| = (-1-\lambda)(4-\lambda)+6 = -4+\lambda-4\lambda+\lambda^2+6$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

The eigen values are $\lambda_1 = 1$, $\lambda_2 = 2$

The Eigen vector at $\lambda_1 = 1$

$$AX_1 = \lambda_1 X_1$$

$$(A-\lambda_1 I)X_1 = 0$$

$$(A-\lambda_1 I)X_1 = \begin{pmatrix} -1-1 & 3 \\ -2 & 4-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 3x_2 = 0$$

Let $x_1 = 3a$, then $x_2 = 2a$

$$X_1 = \begin{pmatrix} 3a \\ 2a \end{pmatrix}$$

The Eigen vector at $\lambda_2 = 2$

$$AX_2 = \lambda_2 X_2$$

$$(A-\lambda_2 I)X_2 = 0$$

$$(A-\lambda_2 I)X_2 = \begin{pmatrix} -1-2 & 3 \\ -2 & 4-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 3x_2 = 0$$

Let $x_1 = a$, then $x_2 = a$

$$X_2 = \begin{pmatrix} a \\ a \end{pmatrix}$$

Sheet 11

No.1-b

$$A = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 7 \end{pmatrix}$$

Solution:

$$A - \lambda I = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 7 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 7 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & 0 & 0 \\ -2 & 1-\lambda & 0 \\ 5 & 3 & 7-\lambda \end{pmatrix}$$

The characteristic equation:

$$|A - \lambda I| = (4-\lambda)(1-\lambda)(7-\lambda) = 0$$

The eigen values are $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 7$

The Eigen vector at $\lambda_1 = 1$

$$AX_1 = \lambda_1 X_1$$

$$(A - \lambda_1 I)X_1 = 0$$

$$(A-\lambda_1 I)X_1=\begin{pmatrix}4-1 & 0 & 0\\ -2 & 1-1 & 0\\ 5 & 3 & 7-1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$=\begin{pmatrix}3 & 0 & 0\\ -2 & 0 & 0\\ 5 & 3 & 6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$3x_1=0$

$5x_1+3x_2+6x_3=0$

$x_1=0$

Let $x_3=a$, then $x_2=-2a$

$$X_1=\begin{pmatrix}0\\ -2a\\ a\end{pmatrix}$$

The Eigen vector at $\lambda_2=4$

$$AX_2=\lambda_2 X_2$$

$(A-\lambda_2 I)X_2=0$

$$(A-\lambda_2 I)X_2=\begin{pmatrix}4-4 & 0 & 0\\ -2 & 1-4 & 0\\ 5 & 3 & 7-4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\ -2 & -3 & 0\\ 5 & 3 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$-2x_1-3x_2=0$

$5x_1+3x_2+3x_3=0$

Let $x_1=3a$, then $x_2=-2a$

$$5(3a)+3(-2a)+3x_3=0$$
$$x_3=-3a$$

$$X_2=\begin{pmatrix}3a\\-2a\\-3a\end{pmatrix}$$

The Eigen vector at $\lambda_3 = 7$

$$AX_3=\lambda_3X_3$$

$$(A-\lambda_3I)X_3=0$$

$$(A-\lambda_3I)X_3=\begin{pmatrix}4-7 & 0 & 0\\-2 & 1-7 & 0\\5 & 3 & 7-7\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 0 & 0\\-2 & -6 & 0\\5 & 3 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$-3x_1=0$$
$$-2x_1-6x_2=0$$
$$x_1=0$$

Then, $x_2=0$, Let $x_3=a$

$$X_3=\begin{pmatrix}0\\0\\a\end{pmatrix}$$

No.1-c

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Solution:

$$A - \lambda I = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{pmatrix}$$

The characteristic equation:

$$|A - \lambda I| = (4-\lambda)(1-\lambda)^2 + 1\big(0 + 2(1-\lambda)\big)$$

$$= (4-\lambda)(1-\lambda)^2 + 2(1-\lambda)$$

$$= (1-\lambda)\big((4-\lambda)(1-\lambda) + 2\big)$$

$$= (1-\lambda)\big(\lambda^2 - 5\lambda + 6\big)$$

$$= (1-\lambda)(\lambda-2)(\lambda-3)$$

The eigen values are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

The Eigen vector at $\lambda_1 = 1$

$$AX_1 = \lambda_1 X_1$$

$$(A - \lambda_1 I)X_1 = 0$$

$$(A-\lambda_1 I)X_1=\begin{pmatrix}4-1 & 0 & 1\\ -2 & 1-1 & 0\\ -2 & 0 & 1-1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$=\begin{pmatrix}3 & 0 & 1\\ -2 & 0 & 0\\ -2 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$3x_1+x_3=0$

$-2x_1=0$

$x_1=0$, Then $x_3=0$

Let $x_2=a$

$$X_1=\begin{pmatrix}0\\ a\\ 0\end{pmatrix}$$

The Eigen vector at $\lambda_2=2$

$$AX_2=\lambda_2 X_2$$

$(A-\lambda_2 I)X_2=0$

$$(A-\lambda_1 I)X_1=\begin{pmatrix}4-2 & 0 & 1\\ -2 & 1-2 & 0\\ -2 & 0 & 1-2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$=\begin{pmatrix}2 & 0 & 1\\ -2 & -1 & 0\\ -2 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$2x_1+x_3=0$

$-2x_1-x_2=0$

Let $x_1 = a$

Then $x_3 = -2a$

and

$x_2 = -2a$

$$X_2 = \begin{pmatrix} a \\ -2a \\ -2a \end{pmatrix}$$

The Eigen vector at $\lambda_2 = 3$

$$AX_3 = \lambda_3 X_3$$

$$(A - \lambda_3 I)X_3 = 0$$

$$(A - \lambda_3 I)X_3 = \begin{pmatrix} 4-3 & 0 & 1 \\ -2 & 1-3 & 0 \\ -2 & 0 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_3 = 0$$

$$-2x_1 - 2x_2 = 0$$

Let $x_1 = a$

Then $x_3 = -a \quad x_2 = -a$

$$X_3 = \begin{pmatrix} a \\ -a \\ -a \end{pmatrix}$$

Cayley-Hamilton theorem

Every square matrix satisfies its characteristics equation. i.e . put $\lambda = A$

Say, If the characteristics equation is

$$a_1\lambda^2 + a_2\lambda + a_3 = 0$$

Then

$$a_1A^2 + a_2A + a_3I = 0$$

No.2-a

$$A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$$

Solution:

$$A - \lambda I = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{pmatrix}$$

The characteristic equation:

$$|A - \lambda I| = (-1-\lambda)(4-\lambda) + 6 = -4 + \lambda - 4\lambda + \lambda^2 + 6$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Put $\lambda = A$

$$A^2 - 3A + 2I = 0$$

Multiply by A^{-1} from the left

$$A^{-1}A^2 - 3A^{-1}A + 2A^{-1}I = 0$$

$$A - 3I + 2A^{-1} = 0$$

$$A^{-1} = \frac{1}{2}(3I - A)$$

$$A^{-1} = \frac{1}{2}(3I - A) = \frac{1}{2}\left(\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$$

No.2-b

$$A = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 7 \end{pmatrix}$$

Solution:

$$A - \lambda I = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 7 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 7 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & 0 & 0 \\ -2 & 1-\lambda & 0 \\ 5 & 3 & 7-\lambda \end{pmatrix}$$

The characteristic equation:

$$|A - \lambda I| = (4-\lambda)(1-\lambda)(7-\lambda)$$

$$= (\lambda^2 - 5\lambda + 4)(7-\lambda)$$

$$= \lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$$

Put $\lambda = A$

$$A^3 - 12A^2 + 39A - 28I = 0$$

Multiply by A^{-1} from the left

$$A^{-1}A^3 - 12A^{-1}A^2 + 39A^{-1}A - 28A^{-1}I = 0$$

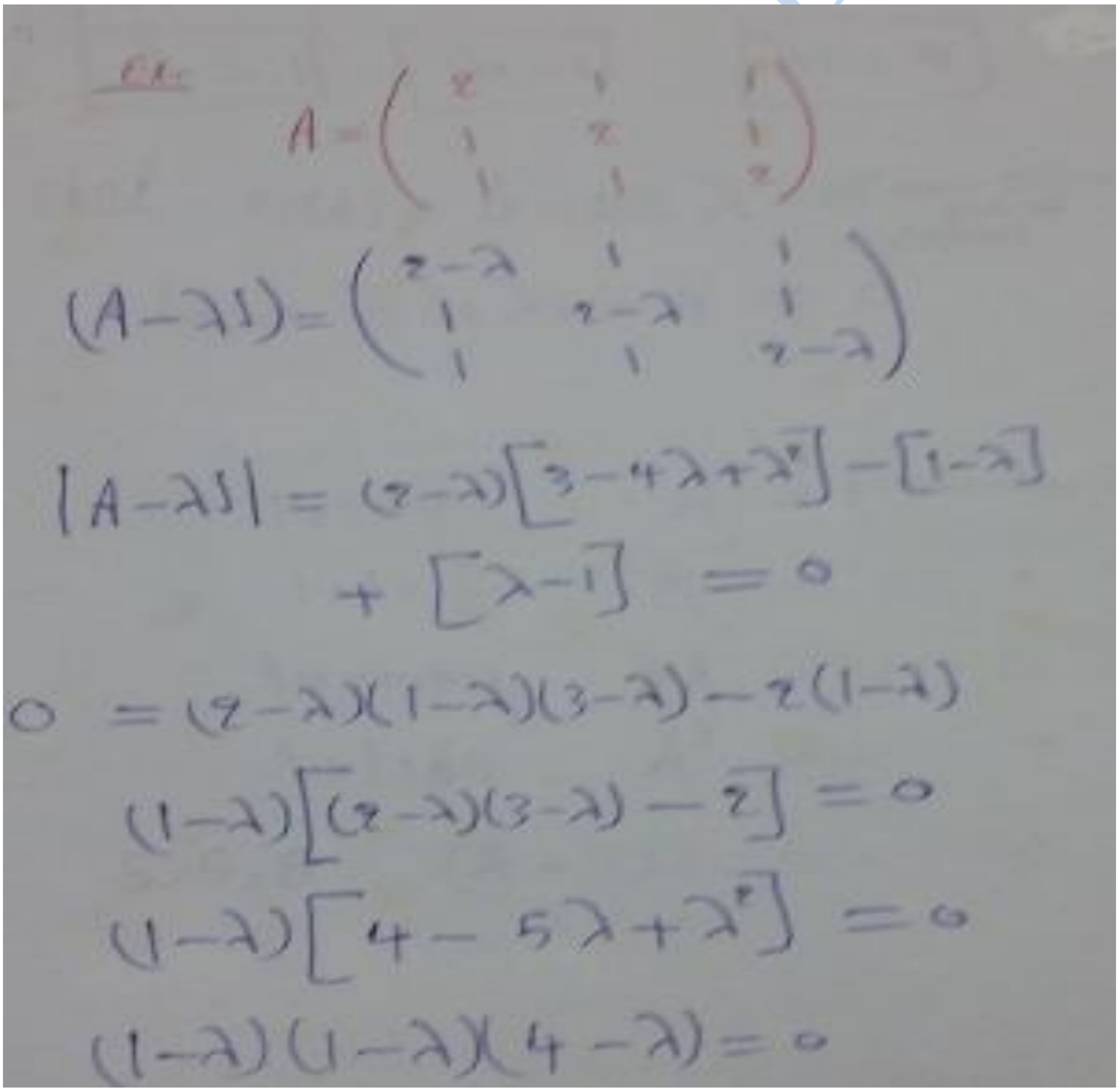
$$A^2 - 12A + 39I - 28A^{-1} = 0$$

$$A^{-1} = \frac{1}{28}\left(A^2 - 12A + 39I\right)$$

$$A^2 = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 7 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 7 \end{pmatrix} = \begin{pmatrix} 16 & 0 & 0 \\ -10 & 1 & 0 \\ 49 & 24 & 49 \end{pmatrix}$$

$$A^{-1} = \frac{1}{28}\left(\begin{pmatrix} 16 & 0 & 0 \\ -10 & 1 & 0 \\ 49 & 24 & 49 \end{pmatrix} - \begin{pmatrix} 48 & 0 & 0 \\ -24 & 12 & 0 \\ 60 & 36 & 84 \end{pmatrix} + \begin{pmatrix} 39 & 0 & 0 \\ 0 & 39 & 0 \\ 0 & 0 & 39 \end{pmatrix}\right)$$

$$= \frac{1}{28}\begin{pmatrix} 7 & 0 & 0 \\ 14 & 28 & 0 \\ -11 & -12 & 4 \end{pmatrix}$$



Ex.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$(A-\lambda I) = \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$|A-\lambda I| = (2-\lambda)[3-4\lambda+\lambda^2] - [1-\lambda] + [\lambda-1] = 0$$

$$0 = (2-\lambda)(1-\lambda)(3-\lambda) - 2(1-\lambda)$$

$$(1-\lambda)[(2-\lambda)(3-\lambda) - 2] = 0$$

$$(1-\lambda)[4 - 5\lambda + \lambda^2] = 0$$

$$(1-\lambda)(1-\lambda)(4-\lambda) = 0$$

$\boxed{\lambda_1 = 1}$ $\boxed{\lambda_2 = 1}$ $\boxed{\lambda_3 = 4}$

check $tr(A) = 6 = \sum \lambda = \sum$ main Diagonal Elements [Trace]

$|A| = \lambda_1 \lambda_2 \lambda_3 = 4 \neq 0$

$1, 1, 4$	A	لـ	Eigenvalues *
$1, 1, \frac{1}{4}$	A^{-1}	لـ	,, ,, *
$10, 10, 40$	$10A$	لـ	,, ,, *
$1, 1, 64$	A^3	لـ	,, ,, ,,
$5, 5, 8$	$(A+4I)$	لـ	,, ,, *

c/c's $(1-\lambda)(4-5\lambda+\lambda^2)=0$

$$4-9\lambda+6\lambda^2-\lambda^3=0$$

$$\lambda^3-6\lambda^2+9\lambda-4=0$$

$\lambda \rightarrow A$ $\quad A^3-6A^2+9A-4I=0$

$*A^{-1}$ $\quad A^2-6A+9I-4A^{-1}=0$

$$A^{-1}=\frac{1}{4}\left[A^2-6A+9I\right]$$

$$\rightarrow A^3=6A^2-9A+4I$$