CHAPTER (1)

VECTOR ANALYSIS

Scalar quantities:

The scalar quantity has only magnitude such as temperature, time and energy…

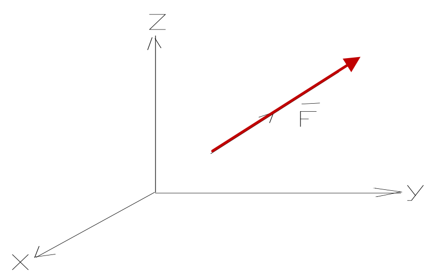
Vector:

A vector is a quantity that has magnitude and direction such as velocity, force and weight…

$$\vec{F} = \hat{a}_{\vec{F}} F$$

F **is the magnitude of the vector** $\vec{F}$

$\hat{a}_{\vec{F}}$ **is the unit vector of vector** $\vec{F}$



Unit vector:

Is a vector of magnitude equal to unity in the direction of the main vector and is defined by:

$$\hat{a}_{\vec{F}} = \frac{\vec{F}}{F}$$

Example:

Determine the unit vector $\hat{a}_{\vec{F}}$ of the vector $\vec{F}$

$$\overline{F} = 2\hat{x} + 5\hat{y} + 6\hat{z}$$

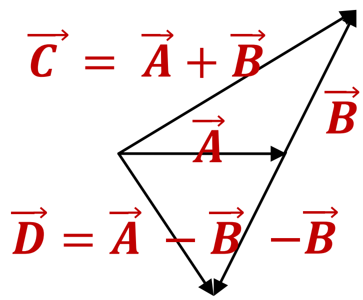
$$\hat{a}_F = \frac{\overline{F}}{|F|} \quad , \quad |F| = \sqrt{(2)^2 + (5)^2 + (6)^2} = \sqrt{4 + 25 + 36} = \sqrt{65}$$

$$\hat{a}_F = \frac{2}{\sqrt{65}}\hat{x} + \frac{5}{\sqrt{65}}\hat{y} + \frac{6}{\sqrt{65}}\hat{z}$$

Vector addition and subtraction:

Graphically:

The sum of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$ that begins at the start of $\vec{A}$ and ends at the arrow of vector $\vec{B}$.



If we express $\vec{A}$ and $\vec{B}$ in components as

$$\overline{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\overline{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

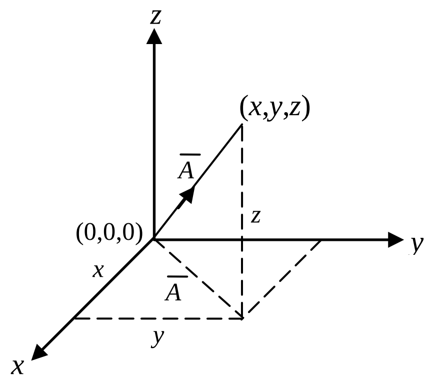
$$\overline{A} + \overline{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z} = \overline{C}$$

$$\overline{A} - \overline{B} = (A_x - B_x)\hat{x} + (A_y - B_y)\hat{y} + (A_z - B_z)\hat{z} = \overline{D}$$

$$\hat{a}_C = \frac{\overline{C}}{|C|} = \frac{(A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}}{\sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}}$$

Position vector:

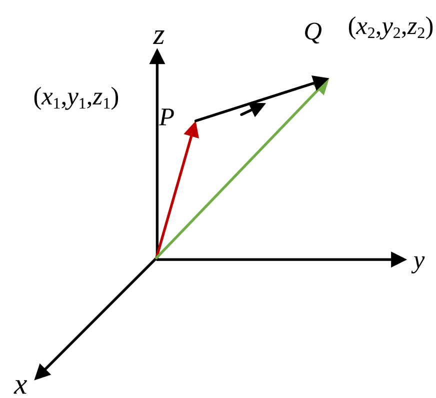
Is defined as the directed distance from the origin to a coordinate point in space.



$$\vec{R}_P = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z$$

Distant vector:

Is defined as the directed distance between two coordinate points in space.



$$\vec{R}_Q = \vec{R}_P + \vec{R}_{PQ}$$

$$\vec{R}_{PQ} = \vec{R}_Q - \vec{R}_P$$

$$\vec{R}_{PQ} = \hat{a}_x(x_2 - x_1) + \hat{a}_y(y_2 - y_1) + \hat{a}_z(z_2 - z_1)$$

$\hat{a}_{\vec{R}_{PQ}}$

$$= \frac{\hat{a}_x(x_2 - x_1) + \hat{a}_y(y_2 - y_1) + \hat{a}_z(z_2 - z_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Vector multiplication:

i-A scalar times a vector:

If we multiply a vector by a scalar quantity

If

$$\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$

Then

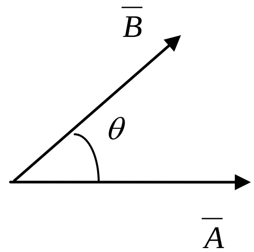
$$m\vec{A} = \hat{a}_x(mA_x) + \hat{a}_y(mA_y) + \hat{a}_z(mA_z)$$

$$|m\vec{A}| = \sqrt{(mA_x)^2 + (mA_y)^2 + ((mA_z)^2}$$

$$|m\vec{A}| = m\sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2} = m|\vec{A}|$$

ii-Dot product (Scaler Product):

$$\vec{A}\,.\,\vec{B} = A\,B\,cos\theta$$



Where: θ is the smallest angle between $\vec{A}$ and $\vec{B}$

$$\theta = cos^{-1}\left(\frac{\vec{A}\,.\,\vec{B}}{A\,B}\right)$$

Projection of $\vec{B}$ on $\vec{A}$ $= B\,cos\theta = \frac{\vec{A}\,.\vec{B}}{A} = \vec{B}.\hat{a}_{\vec{A}}$

Projection of $\vec{A}$ on $\vec{B}$ $= A\,cos\theta = \frac{\vec{A}\,.\vec{B}}{B} = \vec{A}.\hat{a}_{\vec{B}}$

$$\hat{a}_x\,.\,\hat{a}_x = 1, \quad \hat{a}_x\,.\,\hat{a}_y = 0, \quad \hat{a}_x\,.\,\hat{a}_z = 0, \quad and \quad \hat{a}_y\,.\,\hat{a}_z = 0$$

$$\vec{A}\,.\,\vec{B} = (\hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z).(\hat{a}_x B_x + \hat{a}_y B_y + \hat{a}_z B_z)$$

$$\vec{A}\,.\,\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Example:

$$\overline{A} = 2\hat{x} + 3\hat{y} + 10\hat{z}$$

$$\overline{B} = 3\hat{x} + 5\hat{y} + 3\hat{z}$$

Find the angle between $\vec{A}$ and $\vec{B}$

Solution:

$$\theta = \cos^{-1}\left(\frac{\vec{A}\,.\,\vec{B}}{A\,B}\right)$$

$$\left|\overline{A}\right| = \sqrt{2^2 + 3^2 + 10^2} = \sqrt{13}$$

$$\left|\overline{B}\right| = \sqrt{3^2 + 5^2 + 3^2} = \sqrt{43}$$

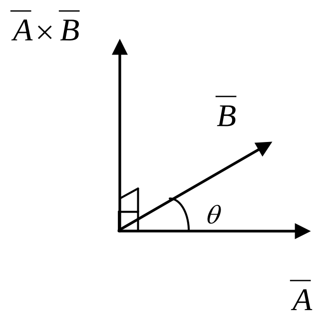
$$\overline{A} \cdot \overline{B} = 6 + 15 + 30 = 51$$

$$\therefore \theta = \cos^{-1}\left(\frac{51}{\sqrt{113}\sqrt{43}}\right)$$

iii- Cross product (vector product):

$$\vec{A}\ x\ \vec{B} = A\ B\ sin\theta\ \hat{a}_n$$

Where: θ is the smallest angle between $\vec{A}$ and $\vec{B}$



$$\theta = sin^{-1}\left(\frac{|\vec{A}\ x\ \vec{B}|}{A\ B}\right)$$

$$\vec{A}\ x\ \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

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$\hat{a}_x\ x\ \hat{a}_x = 0,$

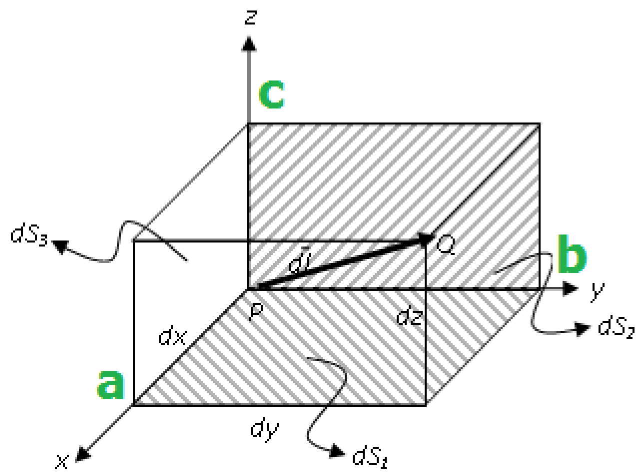
$\hat{a}_x\ x\ \hat{a}_y = \hat{a}_z,$

$\hat{a}_y\ x\ \hat{a}_z = \hat{a}_x,$

$\hat{a}_z\ x\ \hat{a}_x = \hat{a}_y$

Coordinate Systems

i-Rectangular or Cartesian Coordinates (x,y,z):



x = 0 is the equation of a plane parallel to y-z plane at x = 0

y = b is the equation of a plane parallel to x-z plane at y = b

z = 0 is the equation of a plane parallel to x-y plane at z = 0

x = 0 & y = b is the equation of a straight line parallel to z-axis at (0,b,z)

x = 0 & z = 0 is the equation of a straight line parallel to y-axis at (0,y,0)

$y = b$ & $z = 0$ } is the equation of a straight line parallel to x-axis at (x,b,0)

Differential length

$x \longrightarrow dx \longrightarrow h_1 = 1$

$y \longrightarrow dy \longrightarrow h_2 = 1$

$z \longrightarrow dz \longrightarrow h_3 = 1$

Differential Vector Area

$$\overrightarrow{ds_3} = (dz\, dx)\hat{a}_y$$

Differential Volume

$$dv = dx\, dy\, dz$$

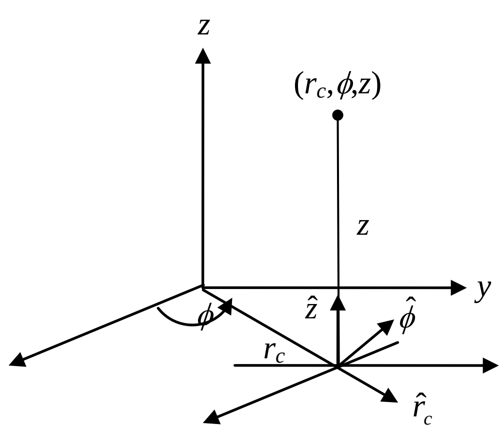
Unit Vectors

$\hat{a}_x$

$\hat{a}_y$ are constant unit vectors

$\hat{a}_z$

ii- Cylindrical coordinates (r_c , φ , z):



$$r_c^2 = x^2 + y^2$$

$$x = r_c \cos\varphi$$

$$y = r_c \sin\varphi$$

$$\tan\varphi = y/x$$

r_c = constant a   is the equation of a cylinder of radius a

φ = constant φ_o is the equation of a plane making angle φ_o with the plane x-z plane

z = constant z_o is the equation of a plane parallel to x-y plane at $z = z_o$

Differential length

r_c	→	dr_c	→	$h_1 = 1$
φ	→	$r_c\, d\varphi$	→	$h_2 = r_c$
z	→	dz	→	$h_3 = 1$

$$\overrightarrow{dl} = \hat{a}_{r_c} dr_c + \hat{a}_\varphi r_c d\varphi + \hat{a}_z dz$$

Differential Vector Area

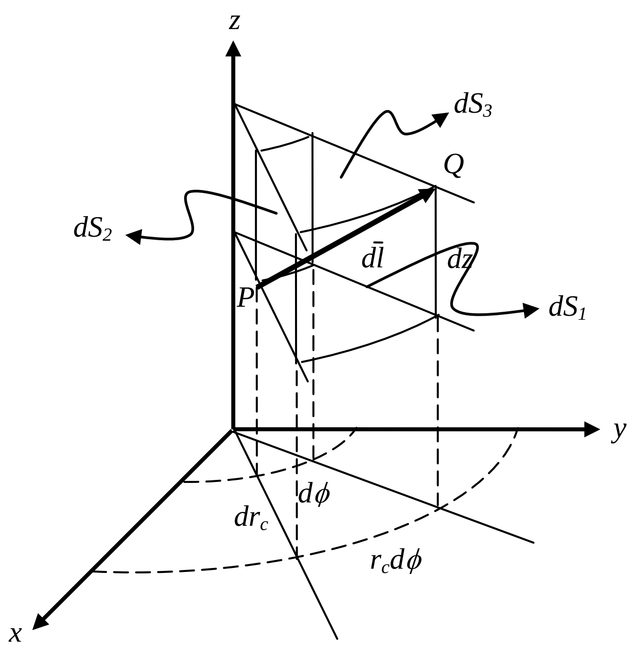
$$\overrightarrow{ds_1} = (r_c d\varphi\, dz)\hat{a}_{r_c}$$

$$\overrightarrow{ds_2} = (dr_c\, dz)\hat{a}_\varphi$$

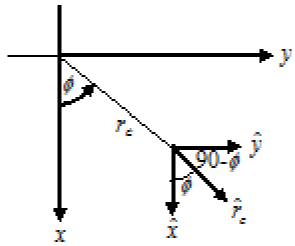
$$\overrightarrow{ds_3} = (r_c dr_c d\varphi)\hat{a}_z$$

Differential Volume

$$dv = r_c dr_c d\varphi\ dz$$



Unit Vectors



$$\hat{a}_x . \hat{a}_{r_c} = cos\varphi$$

$$\hat{a}_y . \hat{a}_{r_c} = cos(90^o - \varphi) = sin\varphi$$

$$\hat{a}_z . \hat{a}_{r_c} = 0$$

$$\hat{a}_\varphi . \hat{a}_{r_c} = 0$$

$$\hat{a}_{r_c} = \hat{a}_x \, cos\varphi + \hat{a}_y \, sin\varphi$$

$$\hat{a}_\varphi = -\hat{a}_x \, sin\varphi + \hat{a}_y \, cos\varphi$$

$$\hat{a}_z = \hat{a}_z$$

$\hat{a}_{r_c}$ *and* $\hat{a}_\varphi$ are not constant unit vectors.
They are function of φ, so they cannot be gotten outside any integral with respect to φ.

Example: **Find the vector directed:**

1-from the point P_{car} (5,10,3) to the point Q_{car} (8,6,20)

2- from the point B_{cy} $(5,30^o,10)$ to point H_{cy} $(6,60^o,20)$

Solution:

$$\overline{PQ} = (8-5)\hat{x} + (6-10)\hat{y} + (10-3)\hat{z}$$

$$= 3\hat{x} - 4\hat{y} + 7\hat{z}$$

Point B:

$$x_1 = r_c \cos\phi = 5\times\cos 30 = \frac{5\sqrt{3}}{2}$$

$$y_1 = r_c \sin 30 = 5\times\sin 30 = 2.5$$

$$z_1 = 10$$

Point H:

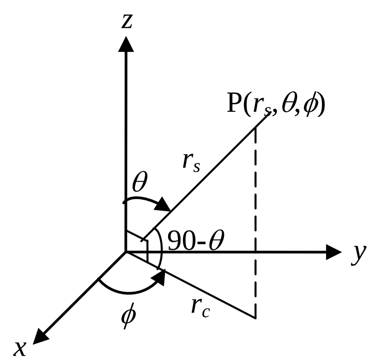
$$x_2 = r_c \cos 60 = 6\times\frac{1}{2} = 3$$

$$y_2 = 6\cdot\sin 60 = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$$

$$z_2 = 20$$

$$\therefore \overline{BH} = \left(3-\frac{5\sqrt{3}}{2}\right)\hat{x} + \left(3\sqrt{3}-2.5\right)\hat{y} + (20-10)\hat{z}$$

iii- Spherical Coordinates (r_s, θ, ϕ):



$$z = r_s \cos\theta$$

$$r_s^2 = r_c^2 + z^2 = x^2 + y^2 + z^2$$

r_s = constant a — **is the equation of a sphere of radius a**

θ = constant θ_o — **is the equation of a cone of angle θ_o**

φ = constant φ_o — **is the equation of a plane making angle φ_o with the plane x-z plane**

Differential length

$r_s \longrightarrow dr_s \longrightarrow h_1 = 1$

$\theta \longrightarrow r_s\, d\theta \longrightarrow h_2 = r_s$

$\varphi \longrightarrow r_s \sin\theta\, d\varphi \longrightarrow h_3 = r_s \sin\theta$

$$\overrightarrow{dl} = \hat{a}_{r_s} dr_s + \hat{a}_\theta r_s d\theta + \hat{a}_\varphi r_s sin\theta d\varphi$$

Differential Vector Area

$$\overrightarrow{ds_1} = \left(r_s^2 sin\theta\, d\theta d\varphi\right)\hat{a}_{r_s}$$

$$\overrightarrow{ds_2} = \left(r_s sin\theta\, dr_s d\varphi\right)\hat{a}_\theta$$

$$\overrightarrow{ds_3} = \left(r_s dr_s d\theta\right)\hat{a}_\varphi$$

Differential Volume

$$dv = r_s^2 sin\theta dr_s d\theta d\varphi$$

Unit Vectors

$$\hat{a}_{r_s} = \hat{a}_{r_c}\, sin\theta + \hat{a}_z\, cos\theta$$

$$= \left(\hat{a}_x\, cos\varphi + \hat{a}_y\, sin\varphi\right) sin\theta + \hat{a}_z\, cos\theta$$

$$\hat{a}_{r_s} = \hat{a}_x\, sin\theta\, cos\varphi + \hat{a}_y\, sin\theta\, sin\varphi + \hat{a}_z\, cos\theta$$

$$\hat{a}_\theta = \hat{a}_{r_c}\, cos\theta - \hat{a}_z\, sin\theta$$

$$= \left(\hat{a}_x\, cos\varphi + \hat{a}_y\, sin\varphi\right) cos\theta - \hat{a}_z\, sin\theta$$

$$\hat{a}_\theta = \hat{a}_x\, cos\theta\, cos\varphi + \hat{a}_y\, cos\theta\, sin\varphi - \hat{a}_z\, sin\theta$$

$$\hat{a}_\varphi = -\hat{a}_x\, sin\varphi + \hat{a}_y\, cos\varphi$$

$$p_p = n_i\, e^{\frac{(E_{Fi} - E_{Fp})}{KT}}\ e^{\frac{(E_{Fn} - E_c)}{KT}}$$

N_A x M_{at} [u] = N_A x M_{at} x 1.66054 x 10^{-24} gm
= 6.022 x 10^{23} M_{at} x 1.66054 x 10^{-24} gm

Xmass of a neutral carbon atom

$$E_{Fp} = E_{Fi} - KT\, ln\left(\frac{p_p}{n_i}\right)$$

$$n_o = N_c\, e^{\frac{(E_{Fn} - E_c)}{KT}}$$

$$P = n$$

$$n = p$$

Vector transformation

Any vector $\overline{A}$ is written in the 3 coordinates systems as follows:

$$\overline{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \Rightarrow cartisian$$

$$\overline{A} = A_{r_c} \hat{r}_c + A_\phi \hat{\phi} + A\ z \Rightarrow cylindrical$$

$$\overline{A} = A_{r_s} \hat{r}_s + A_\theta \hat{\theta} + A_\phi \hat{\phi} \Rightarrow spherical$$

To get any component, we multiply the vector by the unit vector in the direction of the required component:

$$\overline{A} \cdot \hat{x} = A_x$$

$$\overline{A} \cdot \hat{\phi} = A_\phi$$

***Very important note*:** The vector components can be easily obtained as follows:

$$A_x = \overline{A} \cdot \hat{x}, A_y = \overline{A} \cdot \hat{y}, A_z = \overline{A} \cdot \hat{z}, A_{r_c} = A_r \cdot \hat{r}_c$$

$$A_\phi = \overline{A \cdot \hat{\phi}}\,, \overline{A_z} = \overline{A} \cdot \hat{z}\,, \overline{A_{r_s}} = \overline{A \cdot \hat{r}_s}\,, \overline{A_\theta} = \overline{A}.\hat{\theta}\,, \overline{A_\phi} = \overline{A} \cdot \hat{\phi}$$

a- Spherical to Cartesian:

$$\hat{r}_s \cdot \hat{x} = \sin\theta \cos\phi\,, \hat{\theta} \cdot \hat{x} = \cos\theta \cos\phi\,,\ \hat{\phi} \cdot \hat{x} = -\sin\phi$$

$$\hat{r}_s \cdot \hat{y} = \sin\theta \sin\phi\,, \hat{\theta} \cdot \hat{y} = \cos\theta \cdot \sin\phi\,, \hat{\phi} \cdot \hat{y} = \cos\phi$$

$$\hat{r}_s \cdot \hat{z} = \cos\theta\,, \hat{\theta} \cdot \hat{z} = -\sin\theta\,, \hat{\phi} \cdot \hat{z} = zero$$

b- Cylindrical to Cartesian:

$$\hat{\mathbf{r}}_c \cdot \hat{\mathbf{x}} = \cos\phi\,,\ \hat{\phi} \cdot \hat{\mathbf{x}} = -\sin\phi\,,\ \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = \text{zero}$$

$$\hat{\mathbf{r}}_c \cdot \hat{\mathbf{y}} = \sin\phi\,,\ \hat{\phi} \cdot \hat{\mathbf{y}} = \cos\phi\,,\ \hat{\mathbf{z}} \cdot \hat{\mathbf{y}} = \text{zero}$$

$$\hat{\mathbf{r}}_c \cdot \hat{\mathbf{z}} = \text{zero}\,,\ \hat{\phi} \cdot \hat{\mathbf{z}} = \text{zero}\,,\ \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$$

c- Spherical to Cylindrical:

$$\hat{\mathbf{r}}_s \cdot \hat{\mathbf{r}}_c = \sin\theta, \hat{\theta} \cdot \hat{\mathbf{r}}_c = \cos\theta, \hat{\phi} \cdot \hat{\mathbf{r}}_c = \text{Zero}$$

$$\hat{\mathbf{r}}_s \cdot \hat{\phi} = \text{Zero}, \hat{\theta} \cdot \hat{\phi} = \text{Zero}, \hat{\phi} \cdot \hat{\phi} = 1$$

$$\hat{\mathbf{r}}_s \cdot \hat{\mathbf{z}} = \cos\theta, \hat{\theta} \cdot \hat{\mathbf{z}} = -\sin\theta, \hat{\phi} \cdot \hat{\mathbf{z}} = \text{zero}$$

Transformation of a vector from a certain coordinate system to another:

Steps:

1- Write the vector expression in the new coordinate system (general expression).

2- Evaluate the scalar projections into unit vector directions in the new vector directions.

3- Change the variables from old to new coordinate system.

Example:

Transform the vector $\overline{A} = x^2 y\,\hat{x} + y^2 z\,\hat{y} + x^2 z\,\hat{z}$ into cylindrical coordinate.

Solution:

$$\overline{A} = A_{r_c}\hat{r}_c + A_\phi \hat{\phi} + A_z \hat{z}$$

$$A_{r_c} = \overline{A} \cdot \hat{r}_c = (x^2 y\hat{x} + y^2 z\hat{y} + x^2 z\hat{z}) \cdot \hat{r}_c$$

$$A_{r_c} = x^2 y \cos\phi + y^2 z \sin\phi + 0$$

$A_\phi = \overline{A}\cdot\widehat{\phi} = -x^2\sin\phi + y^2 z\cos\phi + 0$

$A_z = \overline{A}\cdot\hat{z} = 0 + 0 + x^2 z$

$\overline{A} = (x^2 y\cos\phi + y^2 z\sin\phi)\widehat{r}_c + (-x^2 y\sin\phi + y^2 z\cos\phi)\widehat{\phi} + (x^2 z)\widehat{z}$

put : $x = r_c\cos\phi$, $y = r_c\sin\phi$, $z = z$

$\therefore \overline{A} = (r_c^3\cos^3\phi\sin\phi + r_c\sin^3\phi)\,\widehat{r}_c + (-r_c^3\cos^2\phi\sin^2\phi + r_c^2\sin^2\phi\cos\phi)\,\widehat{\phi} + (r_c^2\cos^2\phi\cdot z)\,\widehat{z}$

Example

Transform the vector $\overline{A} = r_c\,\hat{r}_c + \cos\phi\,\hat{\phi}$ into:

1- Cartesian
2- Spherical

Solution:

i- Into Cartesian:

$\overline{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$A_x = \overline{A}\cdot\hat{x} = (r_c\,\hat{r}_c + \cos\phi\,\hat{\phi})\cdot\hat{x} = r_c\cos\phi - \cos\phi\sin\phi$

$A_y = \overline{A}\cdot\hat{y} = (r_c\hat{r}_c + \cos\phi\hat{\phi})\cdot\hat{y} = r_c\sin\phi + \cos^2\phi$

$A_z = \overline{A}\cdot\hat{z} = 0$

$\overline{A} = (r_c\cos\phi - \sin\phi\cos\phi)\,\hat{x} + (r_c\sin\phi + \cos^2\phi)\,\hat{y}$

$$\overline{A} = \left(x - \frac{xy}{x^2 + y^2}\right)\hat{x} + \left(y + \frac{x^2}{x^2 + y^2}\right)\hat{y} + 0\,\hat{z}$$

Note: $\cos\phi = \frac{x}{r_c}$, $\sin\phi = \frac{y}{r_c}$

$\therefore \sin\phi\cos\phi = \frac{xy}{r_c^2} = \frac{xy}{x^2 + y^2}$

ii- Into spherical:

$\overline{A} = Ar_s\hat{r}_s + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

$A_{r_s} = \overline{A}\cdot\hat{r}_s = (r_c\,\hat{r}_c + \cos\phi\,\hat{\phi}).\hat{r}_s = r_c\sin\phi$

$A_\theta = \overline{A}\cdot\hat{\theta} = (r_c\,\hat{r}_c + \cos\phi\,\hat{\phi}).\hat{\theta} = r_c\cos\theta$

$A_\phi = \overline{A}\cdot\hat{\phi} = (r_c\,\hat{r}_c + \cos\phi\,\hat{\phi}).\hat{\phi} = \cos\phi$

$\therefore \overline{A} = (r_c\sin\theta)\,\hat{r}_s + r_c\cos\theta\,\hat{\theta} + \cos\phi\,\hat{\phi}$

$= r_s\sin^2\theta\,\hat{r}_s + r_s\sin\theta\cos\theta\,\hat{\theta} + \cos\phi\,\hat{\phi}$

Note: $\frac{x}{\cos\phi} = r_s \sin\theta$

$$x = r_s \cos\phi \sin\theta$$
$$y = r_s \sin\theta \sin\phi$$
$$z = r_s \cos\theta$$

Note:

Let the coordinate system be $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ and $d\bar{l} = h_1 du_1\, \hat{u}_1 + h_2 du_2\, \hat{u}_2 + h_3 du_3\, \hat{u}_3$:

$$d\bar{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \Rightarrow cartisian$$
$$d\bar{l} = dr_c \hat{r}_c + r_c d\phi\,\hat{\phi} + dz\,\hat{z} \Rightarrow cylindrical$$
$$d\bar{l} = dr_s \hat{r}_s + r_s d\theta\,\hat{\theta} + r_s \sin\theta\, d\phi\,\hat{\phi} \Rightarrow spherical$$
$$cartisian: h_1 = h_2 = h_3 = 1$$
$$cylindrical: h_1 = 1,\ h_2 = r_c, h_3 = 1$$
$$spherical: h_1 = 1,\ h_2 = r_s,\ h_3 = r_s \sin\theta$$

i- Grad operated on scalar: $\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\hat{u}_3$

if : $\bar{A} = Au_1\hat{u}_1 + Au_2\hat{u}_2 + Au_3\hat{u}_3$

ii- Div operated on vector: $\nabla \cdot \bar{A} = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial u_1}[h_2 h_3 Au_1] + \frac{\partial}{\partial u_2}[h_1 h_3 Au_2 + \frac{\partial}{\partial u_3}(h_1 h_2 Au_3)]$

iii- $\text{Curl}\,\bar{A} = \nabla \times \bar{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_{u1} & h_2 A_{u2} & h_3 A_{u3} \end{vmatrix}$

Example:

Let $\bar{D} = 5r_c \hat{r}_c + 10\sin\theta\,\hat{\phi} + 20z\,\hat{z}$

Find $\nabla \cdot \bar{D}$, $\nabla \times \bar{D}$

Solution:

$$\nabla \cdot \bar{D} = \frac{1}{r_c}\left[\frac{\partial}{\partial r_c}(5r_c^2) + \frac{\partial}{\partial \phi}(10\sin\theta) + \frac{\partial}{\partial z}(20r_c z)\right]$$

$$\therefore \nabla \cdot \overline{D} = \frac{1}{r_c}[10r_c + 0 + 20r_c] = 30$$

$$\nabla \times \overline{D} = \frac{1}{r_c}\begin{vmatrix} \hat{r}_c & r_c\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r_c} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 5r_c & 10r_c \sin\theta & 20z \end{vmatrix}$$

$$\nabla \times \overline{D} = \frac{1}{r_c}\left(\begin{array}{c} \left[\frac{\partial}{\partial \phi}(20z) - \frac{\partial}{\partial z}(10r_c \sin\theta)\right]\hat{r}_c - r_c\left[\frac{\partial}{\partial r_c}(20z) - \frac{\partial}{\partial z}(5r_c)\right]\hat{\phi} \\ + \left[\frac{\partial}{\partial r_c}(10r_c \sin\theta) - \frac{\partial}{\partial \phi}(5r_c)\right]\hat{z} \end{array} \right)$$