CHAPTER (2)

Electric Charges, Electric Charge Densities and Electric Field Intensity

Charge Configuration

a) Point Charge:

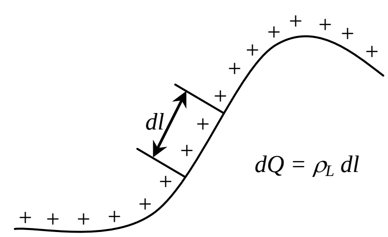
The concept of the point charge is used when the dimensions of an electric charge distribution are very small compared to the distance to the neighboring charges, i.e. the point charge is occupying a very small physical space

b) Distributed Charge

The charge may be distributed along a line, among a surface or among a volume.

(1) Line Charge:

The line charge density ρ_l is defined as the charge per unit length.



$$\rho_l = lim_{\Delta l \to 0} \frac{\Delta Q}{\Delta l} \quad \left[{}^{C}/_{m}\right]$$

$$\rho_l = \frac{dQ}{dl} \quad \left[{}^{C}/_{m}\right]$$

So,

$$dQ = \rho_l \, dl$$

The total charge Q of the line can be determined by

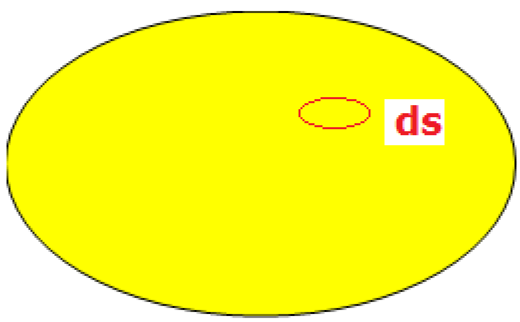
$$Q = \int_0^l dQ = \int_0^l \rho_l \, dl \quad [C]$$

For uniform line charge, where ρ_l is constant

$$Q = \rho_l \, l \quad [C]$$

(2) Surface Charge:

The surface charge density ρ_s is defined as the charge per unit surface area.



$$\rho_s = lim_{\Delta s \to 0} \frac{\Delta Q}{\Delta s} \quad \left[{}^{C}/_{m^2}\right]$$

$$\rho_s = \frac{dQ}{ds} \quad \left[{}^{C}/_{m^2}\right]$$

So,

$$dQ = \rho_s \, ds$$

The total charge Q of the surface can be determined by

$$Q = \iint dQ = \iint \rho_s \, ds$$

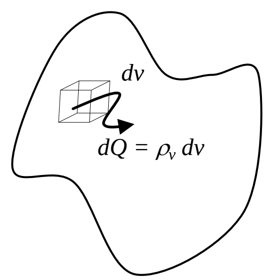
<u>For uniform surface charge</u>, where ρ_s is constant

$$Q = \rho_s S \quad [C]$$

3) Volume charge density (ρ_v)

$$\rho_v = lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v} \quad \left[{}^{C}/_{m^3}\right]$$

$$\rho_v = \frac{dQ}{dv} \quad \left[{}^{C}/_{m^2}\right]$$



So,

$$dQ = \rho_v \, dv$$

The total charge Q of the surface can be determined by

$$Q = \iiint dQ = \iiint \rho_v \, dv$$

<u>For uniform volume charge</u>, where ρ_v is constant

$$Q = \rho_v V \quad [C]$$

Example:

A uniform spherical volume charge density distribution contains a total charge of 10^{-8} C, if the radius of the sphere $=2x10^{-2}$ m. Find ρ_v.

Solution:

$$Q = 10^{-8} C \quad , \quad r = 2*10^{-2} m \quad , \quad V = \frac{4\pi}{3} r^3 = \frac{4\pi}{3}(2*10^{-2})^3 = 8*10^{-6} m^2$$

$$\therefore \rho_v = \frac{Q}{V} = \frac{10^{-8\cdot}}{\frac{4\pi}{3}*8*10^{-8}} = 2.98*10^{-4}\ Cm^{-3}$$

Example:

A non-uniform spherical volume charge density distribution with $\boldsymbol{\rho_v = \frac{k_1}{r_s}}$ C.m^{-3}. Find the total charge contained in the volume of the sphere of radius a [m]

Solution:

$$\boldsymbol{dQ = \rho_v\, dv}$$

$$\boldsymbol{Q = \iiint dQ = \iiint \rho_v\, dv}$$

$$\boldsymbol{= \iiint \frac{k_1}{r_s} r_s^2\, sin\theta\, d\theta\, d\varphi}$$

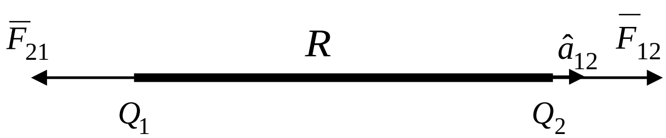
$$Q = k_1 \left[\frac{r_s^2}{2}\right]_0^a [\phi]_0^{2\pi} [-\cos\theta\ ][-\cos\theta\ ]_0^{\pi}$$

$$Q = 2\pi k_1 a^2 \qquad C$$

Coloumb's Law:

Force Between Two Point Charges:

The force between two stationary point charges Q1,Q2 is proportional to the product of the two charges and inversely proportional to the square of the distance R between them.



$$\vec{F}_{12} = \frac{Q_1\, Q_2}{4\pi\varepsilon_o R^2}\,\hat{a}_{\vec{R}_{12}}$$

$$\vec{F}_{21} = \frac{Q_1\, Q_2}{4\pi\varepsilon_o R^2}\,\hat{a}_{\vec{R}_{21}}$$

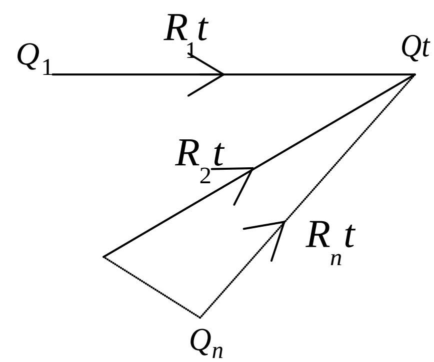
Where: $\hat{a}_{\vec{R}_{12}} = -\hat{a}_{\vec{R}_{21}}$ unit vector from charge Q_1 to Q_2 charge.

and, $\varepsilon_o = \frac{10^{-9}}{36\pi} = 8.85x10^{-12}\ F/m$

Force on Point Charge due to n Point Charges:

Force $\vec{F}_t$ on point charge Q_t due to n point charges can be determined as

$$\vec{F}_t = \sum_{i=1}^{n} \frac{Q_i\, Q_t}{4\pi\varepsilon_o R_{it}^2} \hat{a}_{\vec{R}_{it}} \quad [N]$$



Example:

Find the force $\vec{F}$ in vacuum on a point $Q_1 = 10^{-4}$ C due to a point charge $Q_2 = 2*10^{-5}$ C where Q_1 is centered at point (0,1,2)m and Q_2 at (2,4,5)

Solution:

$$\overline{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2{}_{21}} \hat{a}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^3} \overline{R_{21}}$$

$$\overline{R}_{21} = (0-2)\hat{x} + (1-4)\hat{y} + (2-5)\hat{z}$$

$$= -2\hat{x} - 3\hat{y} - 3\hat{z}$$

$$\therefore R_{21} = \sqrt{4+9+9} = \sqrt{22}$$

$$\hat{a}_{21} = \frac{\overline{R}_{21}}{|R_{21}|} = \frac{-2\hat{x} - 3\hat{y} - 3\hat{z}}{\sqrt{22}}$$

$$\therefore \overline{F} = \frac{(0^{-4} * 2 * 10^{-5}) \cdot (-2\hat{x} - 3\hat{y} - 3\hat{z})}{4\pi * 8.85 * 10^{-12} (\sqrt{22})^2 \cdot \sqrt{22}} \quad N$$

Electric Field Intensity at a Point due to Point Charge Q:

It is a vector force acting on a unit (+ve) charge. The electric field intensity due to a point located at distance R from the charge Q is given by:

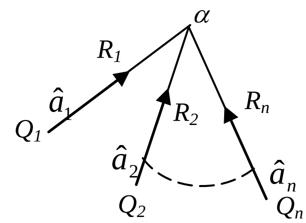
$$\vec{E} = \frac{Q * 1}{4\pi\varepsilon_o R^2} \hat{a}_{\vec{R}} = \frac{Q\vec{R}}{4\pi\varepsilon_o R^3} \; volt/_{meter} \; (V/m)$$

Electric Field Intensity at a Point due to Point Charges Q_1, Q_2,....., Q_n:

If we have a system of charges Q_1, Q_2…Q_n. the total electric field at a point is the vector sum of all fields due to the different charges.

$$\vec{E}_t = \sum_{i=1}^{n} \frac{Q_i \vec{R}}{4\pi\varepsilon_o R^3} \hat{a}_{\vec{R}_{it}} \; [N/C]$$

$$\vec{E}_t = \sum_{i=1}^{n} \frac{Q_i}{4\pi\varepsilon_o R_i^2} \hat{a}_{\vec{R}_i} \; [N/C]$$



Example:

Find the electric field intensity at the point (2,4,5) m, due to a point charge $Q = 2*10^{-5}$ C, located at (0,1,2) m.

Solution:

$$\overline{E} = \frac{Q\,\overline{R}}{4\pi\varepsilon_0 R^3}$$

$$\overline{R} = (2-0)\hat{x} + (4-1)\hat{y} + (5-2)\hat{z}$$

$$= 2\hat{x} + 3\hat{y} + 3z$$

$$R = \sqrt{4+9+9} = \sqrt{22}$$

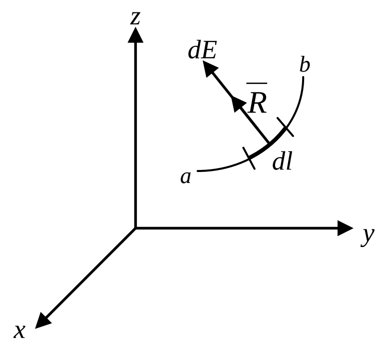
$$\overline{E} = \frac{2*10^{-5}(2\hat{x}+3\hat{y}+3\hat{z})}{4\pi(8.85*10^{-12})(\sqrt{22})^3} = \ldots\,\hat{x} + \ldots\,\hat{y} + \ldots\,\hat{z}$$

Electric Field Intensity at a point p (r_c, ϕ, z) due to line charge

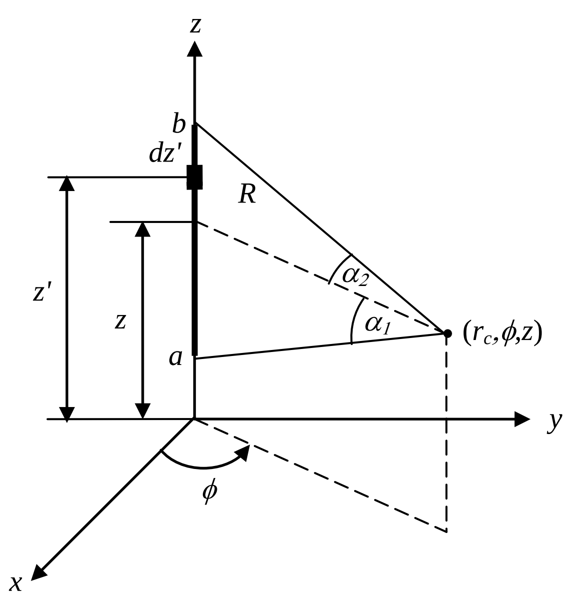
$$d\vec{E} = \frac{dQ\, \hat{a}_{\vec{R}}}{4\pi\varepsilon_o R^2}$$

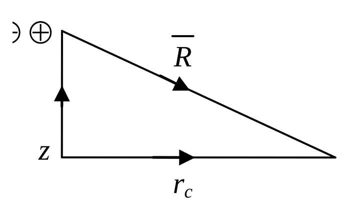
$$= \frac{dQ\, \vec{R}}{4\pi\varepsilon_o R^3}$$

$$\vec{E} = \int_a^b \frac{\rho_l\, dl\, \vec{R}}{4\pi\varepsilon_o R^3}$$



Electric Field Intensity at a Point p (r_c, ϕ, z) due to Uniform Line Charge Along z-Axis





$$d\vec{E} = \frac{dQ\ \hat{a}_{\vec{R}}}{4\pi\varepsilon_o R^2}$$

$$dQ = \rho_l\ dl = \rho_l\ dz'$$

$$\vec{R} = \hat{a}_{r_c} r_c - \hat{a}_z (z' - z)$$

$$R = \sqrt{r_c^2 + (z' - z)^2}$$

$$\hat{a}_{\vec{R}} = \frac{\vec{R}}{R} = \frac{\hat{a}_{r_c} r_c - \hat{a}_z (z' - z)}{\sqrt{r_c^2 + (z' - z)^2}}$$

$$d\vec{E} = \frac{\rho_l\ dz' \left[\hat{a}_{r_c} r_c - \hat{a}_z (z' - z)\right]}{4\pi\varepsilon_o \left[r_c^2 + (z' - z)^2\right]^{3/2}}$$

$$\vec{E} = \frac{\rho_l}{4\pi\varepsilon_o} \int_a^b \frac{\left[\hat{a}_{r_c} r_c - \hat{a}_z \left(z' - z\right)\right]}{\left[r_c^2 + \left(z' - z\right)^2\right]^{3/2}}\ dz'$$

<u>note:</u>

$$\int_a^b \frac{dx}{\left[c^2 + x^2\right]^{3/2}} = \frac{x}{c^2 \left[c^2 + x^2\right]^{1/2}}$$

$$\int_a^b \frac{x\,dx}{\left[c^2+x^2\right]^{3/2}} = \frac{-1}{\left[c^2+x^2\right]^{1/2}}$$

$$\vec{E} = \frac{\rho_l}{4\pi\varepsilon_o}\left[\hat{a}_{r_c} r_c \int_a^b \frac{dz'}{\left[r_c^2+\left(z'-z\right)^2\right]^{3/2}} - \hat{a}_z \int_a^b \frac{\left(z'-z\right)dz'}{\left[r_c^2+\left(z'-z\right)^2\right]^{3/2}}\right]$$

$$\vec{E} = \frac{\rho_l}{4\pi\varepsilon_o}\left[\hat{a}_{r_c} r_c \frac{(z'-z)}{r_c^2\left[r_c^2+(z'-z)^2\right]^{1/2}} + \hat{a}_z \frac{1}{\left[r_c^2+(z'-z)^2\right]^{1/2}}\right]_a^b$$

$$\vec{E} = \frac{\rho_l}{4\pi\varepsilon_o}\left[\frac{\hat{a}_{r_c}}{r_c}\left[\frac{(b-z)}{\left[r_c^2+(b-z)^2\right]^{1/2}} - \frac{(a-z)}{\left[r_c^2+(a-z)^2\right]^{1/2}}\right] + \frac{\hat{a}_z}{r_c}\left[\frac{r_c}{\left[r_c^2+(b-z)^2\right]^{1/2}} - \frac{r_c}{\left[r_c^2+(a-z)^2\right]^{1/2}}\right]\right]_a^b$$

$$\vec{E} = \frac{\rho_l}{4\pi\varepsilon_o}\left[\frac{\hat{a}_{r_c}}{r_c}[\sin\alpha_2 + \sin\alpha_1] + \frac{\hat{a}_z}{r_c}[\cos\alpha_2 - \cos\alpha_1]\right]_a^b$$

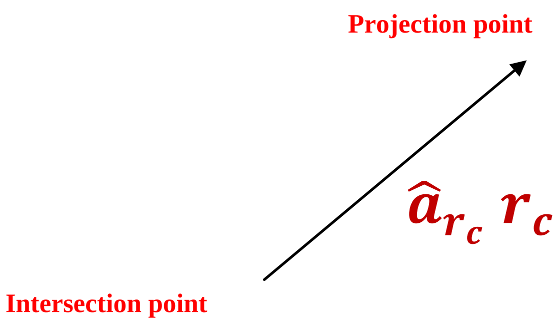
$$\vec{E} = \frac{\rho_l}{4\pi\varepsilon_o r_c}\left[\hat{a}_{r_c}[\sin\alpha_2 + \sin\alpha_1] + \hat{a}_z[\cos\alpha_2 - \cos\alpha_1]\right]$$

Note:

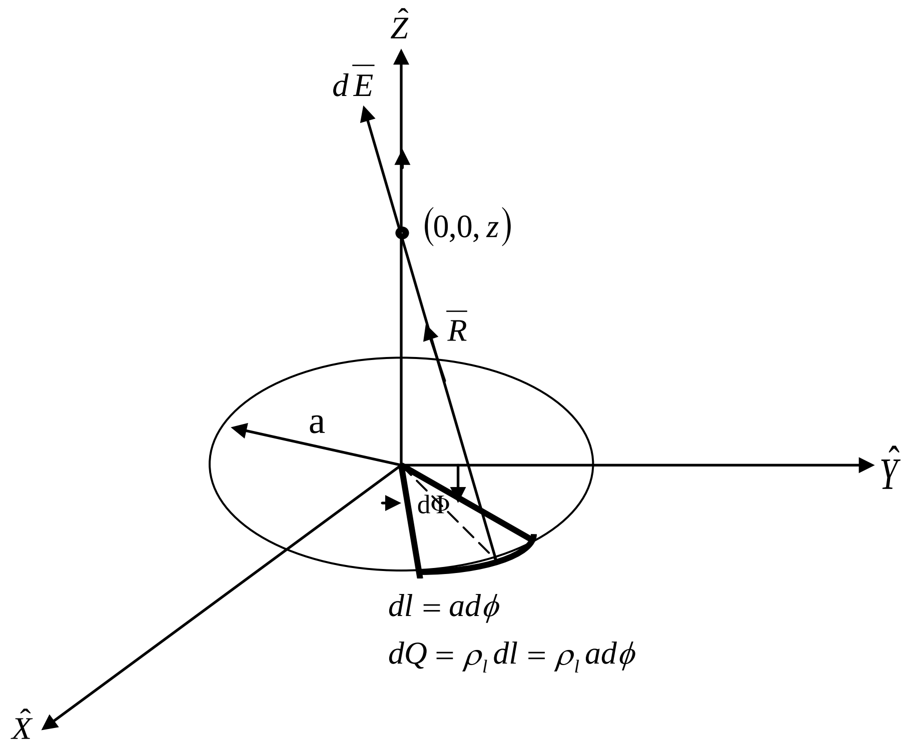
- **For infinite line** $\alpha_2 = \alpha_1 = 90^o$

So,

$$\vec{E} = \hat{a}_{r_c}\frac{\rho_l}{2\pi\varepsilon_o r_c}$$



Electric Field Intensity at a point p (0,0,z) on the axis of a ring charged with uniform ρ_L of radius a centered at the origin and positioned in (x-y) plane



$$d\vec{E} = \frac{dQ\ \hat{a}_{\vec{R}}}{4\pi\varepsilon_o R^2}$$

$$dQ = \rho_l\ dl\ = \rho_l\ r_c^{'} d\varphi^{'}\ = \rho_l\ ad\varphi^{'}$$

$$\vec{R} = -\hat{a}_{r_c} r_c^{'} + \hat{a}_z z = -\hat{a}_{r_c} a + \hat{a}_z z$$

$$R = \sqrt{r_c^{\prime 2} + z^2} = \sqrt{a^2 + z^2}$$

$$\hat{a}_{\vec{R}} = \frac{\vec{R}}{R} = \frac{-\hat{a}_{r_c} a + \hat{a}_z z}{\sqrt{a^2 + z^2}}$$

$$d\vec{E} = \frac{\rho_l \, a d\varphi' \left[-\hat{a}_{r_c} a + \hat{a}_z z\right]}{4\pi\varepsilon_o [a^2 + z^2]^{3/2}}$$

$$\vec{E} = \frac{\rho_l \, a}{4\pi\varepsilon_o} \int_0^{2\pi} \frac{\left[-\hat{a}_{r_c} a + \hat{a}_z z\right]}{[a^2 + z^2]^{3/2}} \, d\varphi'$$

$$\vec{E} = \frac{\rho_l \, a}{4\pi\varepsilon_o [a^2 + z^2]^{3/2}} \left[\int_0^{2\pi} -\hat{a}_{r_c} a \, d\varphi' + \int_0^{2\pi} \hat{a}_z z \, d\varphi'\right]$$

Since, the unit vector $\hat{a}_{r_c}$ is not a constant unit vector and it is a function of φ', and since

$$\hat{a}_{r_c} = \hat{a}_x \cos\varphi' + \hat{a}_y \sin\varphi'$$

So,

$$\vec{E} = \frac{\rho_l \, a}{4\pi\varepsilon_o [a^2 + z^2]^{3/2}} \left[-\int_0^{2\pi} \hat{a}_x \cos\varphi' a \, d\varphi' - \int_0^{2\pi} \hat{a}_y \sin\varphi' a \, d\varphi' + \int_0^{2\pi} \hat{a}_z z \, d\varphi' \right]$$

$$\vec{E} = \frac{\rho_l\, a}{4\pi\varepsilon_o[a^2+z^2]^{3/2}} \int_0^{2\pi} \hat{a}_z z\, d\varphi'$$

$$\vec{E} = \hat{a}_z \frac{\rho_l\, a\, z}{2\varepsilon_o[a^2+z^2]^{3/2}}$$

Example:

A uniform line charge of infinite extent with $\rho_l = 20\, nC/m$ lies on z-axis. Find $\vec{E}$ at (6,8,3) m.

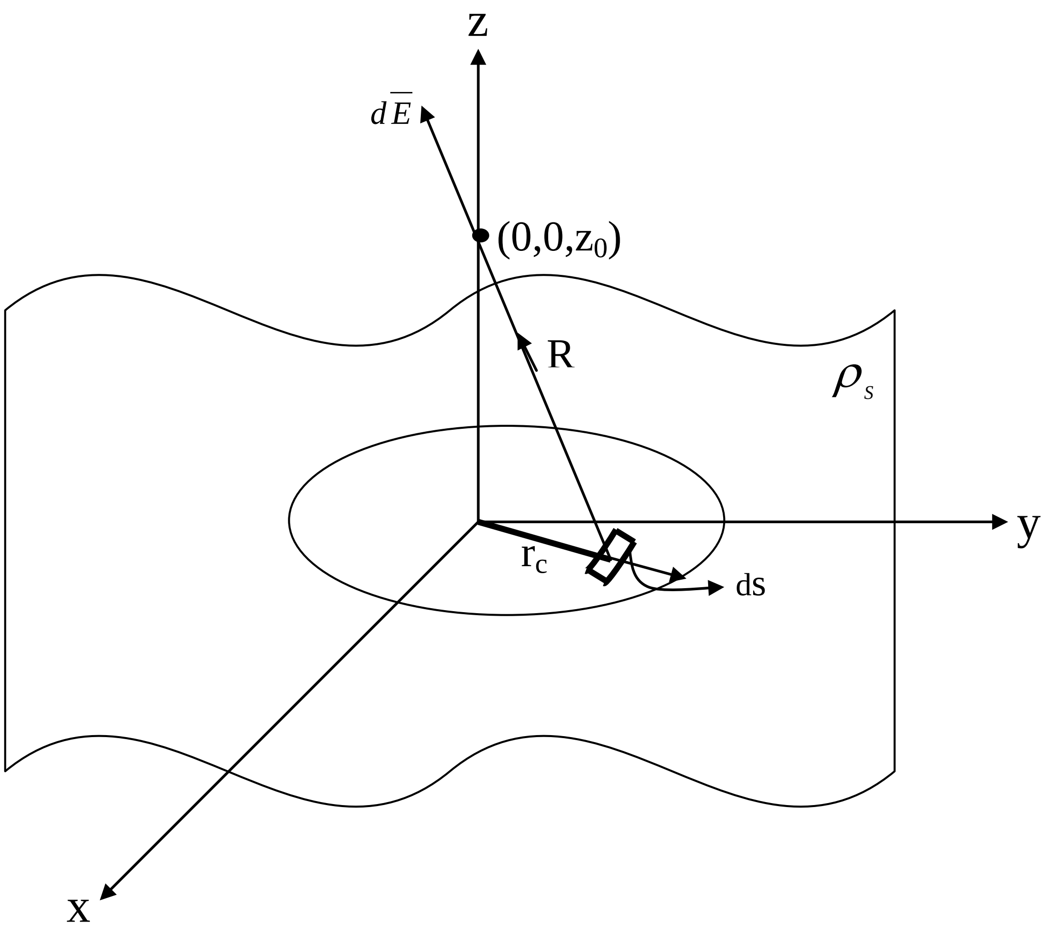
Solution:

$$\overline{E} = \frac{\rho_l}{2\pi\varepsilon_0 r_c}\hat{r}_c$$

$$r_c = \sqrt{x^2+y^2} = \sqrt{6^2+8^2} = 10$$

$$\therefore \overline{E} = \frac{20 * 10^{-9}\hat{r}_c}{2\pi(8.85 * 10^{-2})^{10}\, 10} = 36\hat{r}_c \quad V/m$$

Electric field intensity of a surface charge:



$$d\vec{E} = \frac{dQ\,\hat{a}_{\vec{R}}}{4\pi\varepsilon_o R^2}$$

$$dQ = \rho_s\, ds' = \rho_s\, r'_c dr'_c d\varphi'$$

$$\vec{R} = -\hat{a}_{r_c} r'_c + \hat{a}_z z$$

$$R = \sqrt{r'^2_c + z^2}$$

$$\hat{a}_{\vec{R}} = \frac{\vec{R}}{R} = \frac{-\hat{a}_{r_c} r'_c + \hat{a}_z z}{\sqrt{r'^2_c + z^2}}$$

$$d\vec{E} = \frac{\rho_s \, r'_c dr'_c d\varphi' \left[-\hat{a}_{r_c} r'_c + \hat{a}_z z\right]}{4\pi\varepsilon_o \left[r'^{\,2}_c + z^2\right]^{3/2}}$$

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon_o}\left[\int_0^a \frac{\left[r'^{\,2}_c\right] dr'_c}{\left[r'^{\,2}_c + z^2\right]^{3/2}} \int_0^{2\pi} \left[-\hat{a}_{r_c}\right] d\varphi' + \int_0^a \frac{\left[r'_c\right] dr'_c}{\left[r'^{\,2}_c + z^2\right]^{3/2}} \int_0^{2\pi} \hat{a}_z z \, d\varphi'\right]$$

Since, the unit vector $\hat{a}_{r_c}$ is not a constant unit vector and it is a function of φ', and since

$$\hat{a}_{r_c} = \hat{a}_x \cos\varphi' + \hat{a}_y \sin\varphi'$$

So,

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon_o}\left[\int_0^a \frac{\left[r'^{\,2}_c\right] dr'_c}{\left[r'^{\,2}_c + z^2\right]^{3/2}} \int_0^{2\pi} \left[-\hat{a}_x \cos\varphi' - \hat{a}_y \sin\varphi'\right] d\varphi' + \int_0^a \frac{\left[r'_c\right] dr'_c}{\left[r'^{\,2}_c + z^2\right]^{3/2}} \int_0^{2\pi} \hat{a}_z z \, d\varphi'\right]$$

Then,

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon_o}\left[\int_0^a \frac{[r'_c]dr'_c}{\left[{r'_c}^2 + z^2\right]^{3/2}} \int_0^{2\pi} \hat{a}_z z\, d\varphi'\right]$$

and since,

$$\int_0^a \frac{r'_c dr'_c}{\left[{r'_c}^2+z^2\right]^{3/2}} = \left[\frac{-1}{\left[{r'_c}^2+z^2\right]^{1/2}}\right]_0^a = \left[\frac{-1}{[a^2+z^2]^{1/2}} + \frac{1}{[z^2]^{1/2}}\right]$$

So,

$$\vec{E} = \hat{a}_z \frac{\rho_s}{2\varepsilon_o}\left[\frac{-z}{\left[a^2 + z^2\right]^{1/2}} + \frac{z}{z}\right]$$

$$\vec{E} = \hat{a}_z \frac{\rho_s}{2\varepsilon_o}\left[1 - \frac{z}{\left[a^2 + z^2\right]^{1/2}}\right]$$

For infinite surface $a \rightarrow \infty$

$$\vec{E} = \hat{a}_z \frac{\rho_s}{2\varepsilon_o}$$

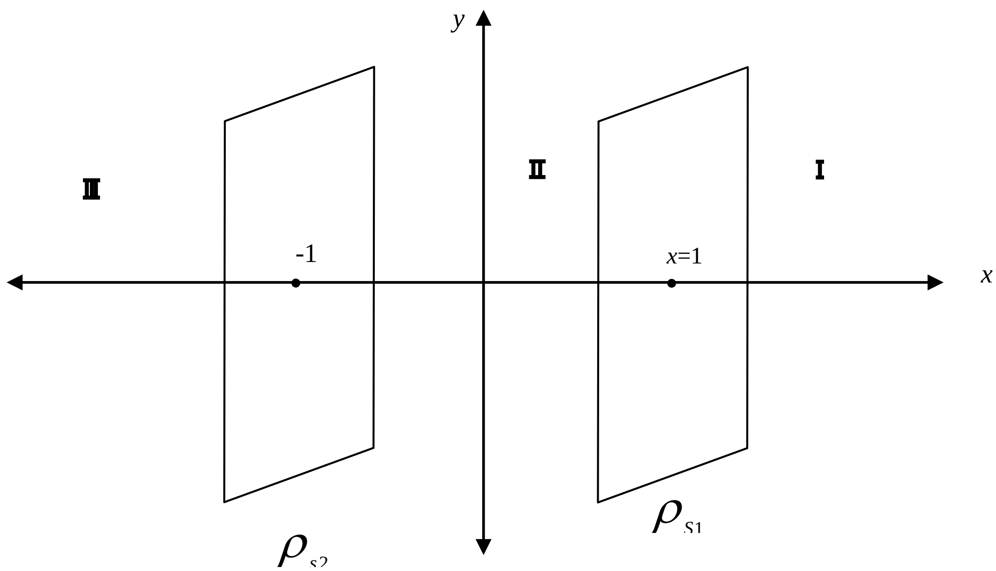
Note:

For infinite surface the electric field $\vec{E} = \hat{a}_z \frac{\rho_s}{2\varepsilon_o}$ **and in direction normal to the surface and out of it.**

Example:

Two infinite uniform sheets of charge ρ_{s1} *and* ρ_{s2} located at $x = \pm 1$ as shown in figure. Find the electric field in all regions.

Solution:



Region 1:

$$\overline{E_1} = \frac{\rho_{s_1}}{2\varepsilon_0}\hat{x} \quad , \quad \overline{E_2} = \frac{\rho_{s_2}}{2\varepsilon_0}\hat{x}$$

$$\therefore \overline{E_t} = \overline{E_1} + \overline{E_2} = \frac{1}{2\varepsilon_0}\left[\rho_{s_1} + \rho_{s_2}\right]\hat{x}$$

Region 2:

$$\overline{E_1} = \frac{\rho_{s_1}}{2\varepsilon_0}(-\hat{x}) \quad , \quad \overline{E_2} = \frac{\rho_{s_2}}{2\varepsilon_0}(\hat{x})$$

$$\therefore \overline{E_t} = \frac{1}{2\varepsilon_0}\left[-\rho_{s_1} + \rho_{s_2}\right]\hat{x}$$

Region 3:

$$\overline{E_1} = -\frac{\rho_{s_1}}{2\varepsilon_0}\hat{x} \quad , \quad \overline{E_2} = -\frac{\rho_{s_2}}{2\varepsilon_0}\hat{x}$$

$$\therefore \overline{E_t} = -\frac{1}{2\varepsilon_0}\left[\rho_{s_1} + \rho_{s_2}\right]\hat{x}$$