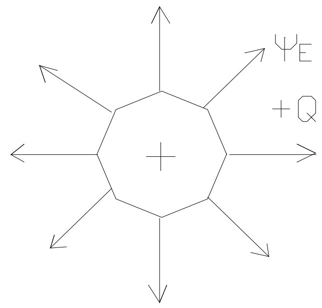
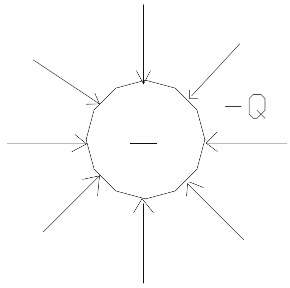
CHAPTER (3)

ELECTRIC FLUX DENSITY

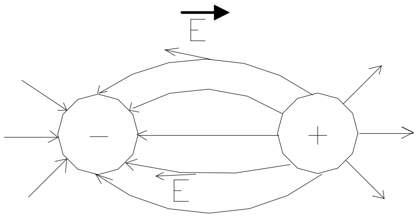
Electric flux (ψ_e):





The electric flux concept is based on the following rules:

1- **Electric flux begins from (+ ve) charge and ends to (-ve) charge**
2- **Electric field at a point is tangent to the electric flux line passing with this point and out wide.**



3- **In the absence of (-ve) charge the electric flux terminates at infinity.**
4- **The magnitude of the electric field at a point is proportional to the magnitude of the electric flux density at this point.**
5- **The number of electric flux lines from a (+ ve) charge Q is equal to Q in SI unit**

$$\psi_e = Q$$

Electric flux density $\overrightarrow{D}$ displacement vector):

In free space, the electric flux density vector $\vec{D}$ is defined as

$$\vec{D} = \hat{a}_n \, lim_{\Delta s \to 0} \frac{\Delta\psi_e}{\Delta s} \quad C\, m^{-2}$$,

Where: $\Delta\psi_e$ equals the number of electric lines that are normal to the surface ΔS

$$\psi_e = \iint \vec{D} \,.\, \overrightarrow{ds}$$

Relation Between $\vec{D}$ and $\vec{E}$ due to Point Charge

If we locate a point charge Q at the origin, the electric flux density $\vec{D}$ can be evaluated by dividing ψ_e by the surface area of the sphere, thus

$$\vec{D} = \hat{a}_{r_s} \frac{\psi_e}{4\pi r_s^2}$$

$$\vec{D} = \hat{a}_{r_s} \frac{Q}{4\pi r_s^2} C\, m^{-2}$$

The expression for $\vec{E}$ on the surface at r_s due to Q, is

$$\vec{E} = \hat{a}_{r_s} \frac{Q}{4\pi \varepsilon_o r_s^2} \quad N\, C^{-1}$$

From the expressions for $\vec{D}$ and $\vec{E}$, it can be seen that

$$\vec{D} = \varepsilon_o \vec{E}$$

The relation between $\vec{D}$ and $\vec{E}$ was derived using a Point charge Q, but also it is valid for general charge distribution,

$$\vec{E} = \iiint \frac{\rho_v \, dv}{4\pi\varepsilon_o R^2} \hat{a}_R$$

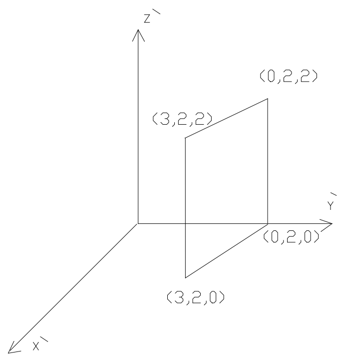
$$\vec{D} = \iiint \frac{\rho_v \, dv}{4\pi R^2} \hat{a}_R$$

From Faraday's experiment, it is found that, ψ_e and thus $\vec{D}$ are independent of the dielectric media in which Q is embedded.

Example:

Find the electric flux ψ_e that passes through the surface shown in the figure. Where:

$$\vec{D} = (y\,\hat{a}_x + x\,\hat{a}_y) x\, 10^{-2}\, C\, m^{-2}$$



Solution

$$\psi_e = \iint \vec{D} \, . \, \vec{ds}$$

$$\psi_e = \int_0^2 \int_0^3 (y\,\hat{a}_x + x\,\hat{a}_y) x\, 10^{-2} . \, \hat{a}_y \, dxdz$$

$$\psi_e = \left[\frac{x^2}{2}\right]_0^3 [z]_0^2 \, x10^{-2} = \left(\frac{9}{2} \, x \, 2\right) x10^{-2} = 9x10^{-2} \, C$$

Gauss's law

As it is stated before, the total electric flux emanating from a charge + Q [C] is equal to Q [C] in the SI units.

The previous statement can be restated by saying that the total electric flux passing through any closed imaginary surface, enclosing the charge Q [C], is equal to Q [C] in the SI units.

Since the charge Q is enclosed by the closed surface, so the charge Q will be named as $Q_{enclosed}$.

Gauss's law states that: the total flux out of a closed surface is equal to the net charges within the surface. This can be written in integral form as:

$$\psi_e = \oiint d\,\psi_e = \oiint \vec{D} \cdot \vec{ds} = Q_{enclosed}$$

Gauss's law is used in order to determine $\vec{D}$ and then $\vec{E}$ by getting $\vec{D}$ outside the closed surface integral. This can be executed by choosing Gaussian surface that satisfies the following conditions, such that $\vec{D}$ be independent of ds variables.

Conditions for Gauss's law:

1- The surface or volume contained charges must has degree of symmetry.

2- $\vec{D}$ must be defined in the surface ($\vec{D} \neq \infty$).

3- $\vec{D}$ must be uniform on the Gaussian surface

4- The Gaussian surface must be identical to the body contained the charge.

Note:

1- Gauss's law is not used for all cases of charges, but it can be used only for the cases where the chosen Gaussian surface satisfy the previous conditions.

2- Gauss's law is used for the following cases:

- Infinite line charges and coaxial charged cylinders
- Infinite charged sheet
- Concentric charged spheres

Example:

Find the electric flux density at a point p(r_c, φ, z) due to an infinite charged line of ρ_l at z-axis.

Solution:

(1) $\oiint \vec{D} \cdot \vec{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

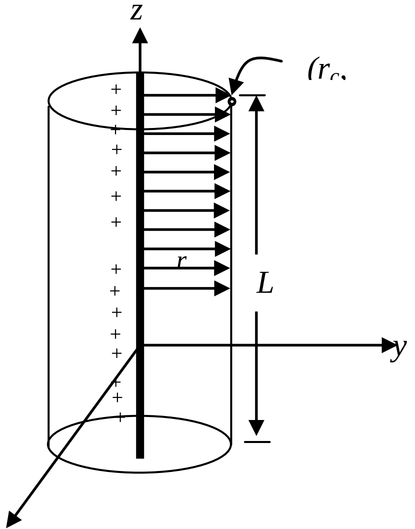
(3) $Q_{enclosed} = \rho_l L$

(4) $\oiint \vec{D} \cdot \vec{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{L} \hat{a}_{r_c} (r_c \, d\varphi \, dz) = 2\pi r_c L D$

(5) $2\pi r_c L D = \rho_l L$

(6) $\vec{D} = \hat{a}_{r_c} \frac{\rho_l}{2\pi r_c} \, C\, m^{-2}$

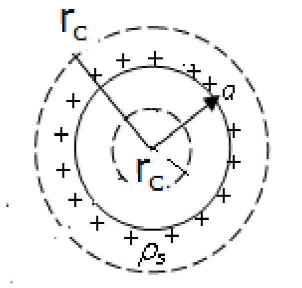
(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_c} \frac{\rho_l}{2\pi r_c \, \varepsilon_o} \; N/C$



Example:

Find $\vec{D}$ and $\vec{E}$ inside and outside a sphere of radius (a) and surface charge density ρ_s.

Solution:



Region 1 $r_s < a$

(1) $\oiint \vec{D} \cdot \overrightarrow{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

(3) $Q_{enclosed} = 0$

(4) $\oiint \vec{D} \cdot \overrightarrow{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} \left(r_s^2 \sin\theta d\theta d\varphi \right) = 4\pi\, r_s^2 D$

(5) $4\pi r_s^2 D = 0$

(6) $\vec{D} = \hat{a}_{r_s}\ 0\ \ C\, m^{-2}$

(7) $\vec{E} = {\vec{D}}/{\varepsilon_o} = \hat{a}_{r_s}\ 0\ \ N\, C^{-1}$

Region 2 $r_s > a$

(1) $\oiint \vec{D} \cdot \overrightarrow{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

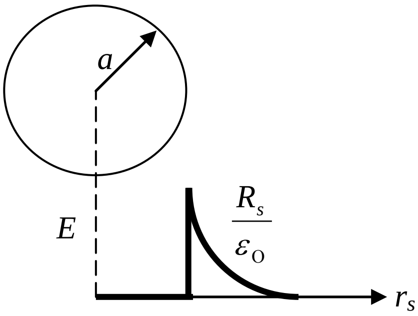
(3) $Q_{enclosed} = 4\pi\, a^2\, \rho_s$

(4) $\oiint \vec{D} \cdot \overrightarrow{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} \left(r_s^2 \sin\theta d\theta d\varphi \right) = 4\pi\, r_s^2 D$

(5) $4\pi r_s^2 D = 4\pi\, a^2\, \rho_s$

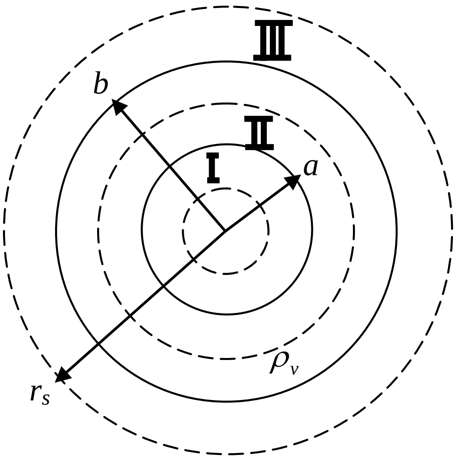
(6) $\vec{D} = \hat{a}_{r_s} \frac{a^2}{r_s^2} \rho_s\ \ C\, m^{-2}$

(7) $\vec{E} = {\vec{D}}/{\varepsilon_o} = \hat{a}_{r_s} \frac{a^2}{\varepsilon_o r_s^2} \rho_s\ \ N\, C^{-1}$



Example:

Find $\vec{D}$ and $\vec{E}$ in all regions for a spherical shell of radii a, b and volume charge density ρ_v



Solution:

Region 1 $r_s < a$

(1) $\oiint \vec{D} \cdot \overrightarrow{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

(3) $Q_{enclosed} = 0$

(4) $\oiint \vec{D} \cdot \overrightarrow{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} \left(r_s^2 \sin\theta \, d\theta \, d\varphi \right) = 4\pi \, r_s^2 D$

(5) $4\pi r_s^2 D = 0$

(6) $\vec{D} = \hat{a}_{r_s} \, 0 \; C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_s} \, 0 \; N\, C^{-1}$

Region 2 $a < r_s < b$

(1) $\oiint \vec{D} \cdot \vec{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

(3) $Q_{enclosed} = \frac{4\pi}{3} (r_s^3 - a^3)\, \rho_v$

(4) $\oiint \vec{D} \cdot \vec{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} \left(r_s^2 \sin\theta\, d\theta\, d\varphi\right) = 4\pi\, r_s^2 D$

(5) $4\pi r_s^2 D = \frac{4\pi}{3} (r_s^3 - a^3)\, \rho_v$

(6) $\vec{D} = \hat{a}_{r_s} \frac{(r_s^3 - a^3)}{3 r_s^2}\, \rho_v\ C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_s} \frac{(r_s^3 - a^3)}{3 \varepsilon_o r_s^2}\, \rho_v\ N\, C^{-1}$

Region 3 $r_s > b$

(1) $\oiint \vec{D} \cdot \vec{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

(3) $Q_{enclosed} = \frac{4\pi}{3} (b^3 - a^3)\, \rho_v$

(4) $\oiint \vec{D} \cdot \vec{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} \left(r_s^2 \sin\theta\, d\theta\, d\varphi\right) = 4\pi\, r_s^2 D$

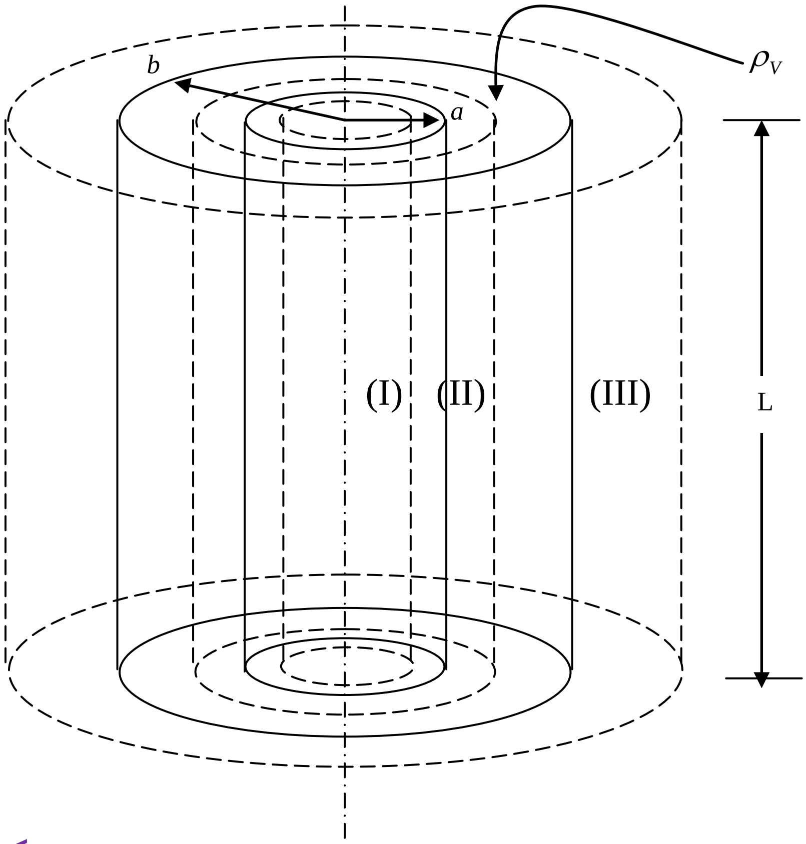
(5) $4\pi r_s^2 D = \frac{4\pi}{3} (b^3 - a^3)\, \rho_v$

(6) $\vec{D} = \hat{a}_{r_s} \frac{(b^3 - a^3)}{3 r_s^2}\, \rho_v\ C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_s} \frac{(b^3 - a^3)}{3 \varepsilon_o r_s^2}\, \rho_v\ N\, C^{-1}$

Example:

In the figure shown, find the electric field intensity in all regions.



Solution

Region 1 $r_c < a$

(1) $\oiint \vec{D} \cdot \overrightarrow{ds} = Q_{enclosed}$

(2) **Choice of Gaussian surface**

(3) $Q_{enclosed} = 0$

(4) $\oiint \vec{D} \cdot \overrightarrow{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{L} \hat{a}_{r_c} (r_c \, d\varphi \, dz) = 2\pi r_c LD$

(5) $2\pi r_c LD = 0$

(6) $\vec{D} = \hat{a}_{r_c} \, 0 \; C \, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_c} \, 0 \quad N \, C^{-1}$

Region 2 $a < r_c < b$

(1) $\oint\!\!\oint \vec{D} \cdot \vec{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

(3) $Q_{enclosed} = \left(\pi r_c^2 - \pi a^2\right) L\, \rho_v$

(4) $\oint\!\!\oint \vec{D} \cdot \vec{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{L} \hat{a}_{r_c} \left(r_c\, d\varphi\, dz\right) = 2\pi r_c LD$

(5) $2\pi r_c LD = \left(\pi r_c^2 - \pi a^2\right) L\, \rho_v$

(6) $\vec{D} = \hat{a}_{r_c} \frac{\left(r_c^2 - a^2\right)}{2r_c} \rho_v \quad C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_c} \frac{\left(r_c^2 - a^2\right)}{2\varepsilon_o r_c} \rho_v \quad N\, C^{-1}$

Region 3 $r_c > b$

(1) $\oint\!\!\oint \vec{D} \cdot \vec{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

(3) $Q_{enclosed} = \left(\pi b^2 - \pi a^2\right) L\, \rho_v$

(4) $\oint\!\!\oint \vec{D} \cdot \vec{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{L} \hat{a}_{r_c} \left(r_c\, d\varphi\, dz\right) = 2\pi r_c LD$

(5) $2\pi r_c LD = \left(\pi b^2 - \pi a^2\right) L\, \rho_v$

(6) $\vec{D} = \hat{a}_{r_c} \frac{\left(\pi b^2 - \pi a^2\right)}{2r_c} \rho_v \quad C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_c} \frac{\left(\pi b^2 - \pi a^2\right)}{2\varepsilon_o r_c} \rho_v \quad N\, C^{-1}$

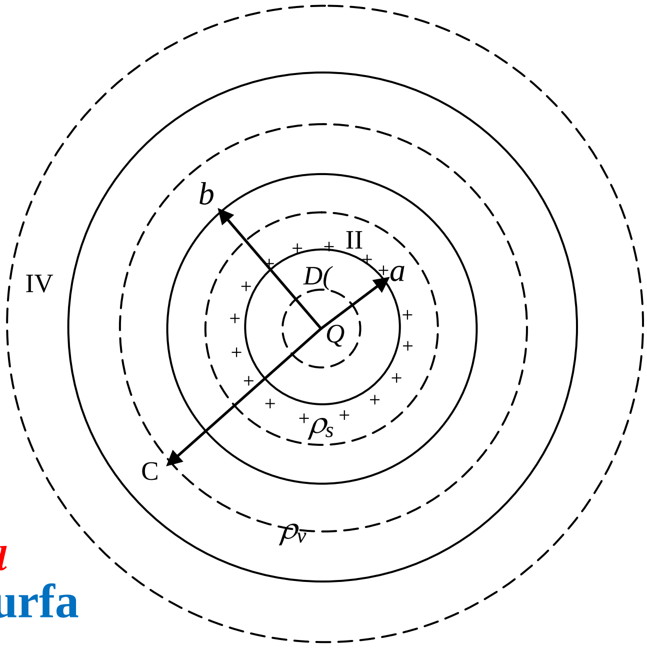
Example:

Find the electric field intensity in all regions for the following charge configurations:

- Point charge Q is located at the center.

- Conducting sphere of radius *a* and of charge ρ_s.

- A volume charge of ρ_v in a spherical shell of radii b, c.



Solution:

Region 1 $r_s < a$

(1) $\oiint \vec{D} \cdot \vec{ds} = Q_{enclosed}$

(2) Choice of Gaussian surfa

(3) $Q_{enclosed} = Q$

(4) $\oiint \vec{D} \cdot \vec{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} \left(r_s^2 \sin\theta \, d\theta \, d\varphi \right) = 4\pi \, r_s^2 D$

(5) $4\pi r_s^2 D = Q$

(6) $\vec{D} = \hat{a}_{r_s} \frac{Q}{4\pi r_s^2} \; C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_s} \frac{Q}{4\pi\varepsilon_o r_s^2} \; N\, C^{-1}$

Region 2 $a < r_s < b$

(1) $\oiint \vec{D} \cdot \vec{ds} = Q_{enclosed}$

(2) Choice of Gaussian surface

(3) $Q_{enclosed} = Q + 4\pi a^2 \rho_s$

(4) $\oiint \vec{D} \cdot \vec{ds} = \vec{D} \cdot \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} \left(r_s^2 \sin\theta \, d\theta \, d\varphi \right) = 4\pi \, r_s^2 D$

(5) $4\pi r_s^2 D = Q + 4\pi a^2 \rho_s$

(6) $\vec{D} = \hat{a}_{r_s} \frac{Q+4\pi a^2 \rho_s}{4\pi r_s^2} \quad C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_s} \frac{Q+4\pi a^2 \rho_s}{4\pi \varepsilon_o r_s^2} \rho_v \; N\, C^{-1}$

Region 3 $b < r_s < c$

(1) $\oiint \vec{D} . \vec{ds} = Q_{enclosed}$

(2) **Choice of Gaussian surface**

(3) $Q_{enclosed} = Q + 4\pi a^2 \rho_s + \frac{4\pi}{3} (r_s^3 - a^3) \rho_v$

(4) $\oiint \vec{D} . \vec{ds} = \vec{D} . \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} (r_s^2 \sin\theta d\theta d\varphi) = 4\pi\, r_s^2 D$

(5) $4\pi r_s^2 D = Q + 4\pi a^2 \rho_s + \frac{4\pi}{3} (r_s^3 - a^3) \rho_v$

(6) $\vec{D} = \hat{a}_{r_s} \frac{Q+4\pi a^2 \rho_s + \frac{4\pi}{3}(r_s^3 - a^3) \rho_v}{4\pi r_s^2} \quad C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_s} \frac{Q+4\pi a^2 \rho_s + \frac{4\pi}{3}(r_s^3 - a^3) \rho_v}{4\pi \varepsilon_o r_s^2} \quad N\, C^{-1}$

Region 4 $r_s > c$

(1) $\oiint \vec{D} . \vec{ds} = Q_{enclosed}$

(2) **Choice of Gaussian surface**

(3) $Q_{enclosed} = Q + 4\pi a^2 \rho_s + \frac{4\pi}{3} (b^3 - a^3) \rho_v$

(4) $\oiint \vec{D} . \vec{ds} = \vec{D} . \int_0^{2\pi} \int_0^{\pi} \hat{a}_{r_s} (r_s^2 \sin\theta d\theta d\varphi) = 4\pi\, r_s^2 D$

(5) $4\pi r_s^2 D = Q + 4\pi a^2 \rho_s + \frac{4\pi}{3} (b^3 - a^3) \rho_v$

(6) $\vec{D} = \hat{a}_{r_s} \frac{Q+4\pi a^2 \rho_s + \frac{4\pi}{3}(b^3 - a^3) \rho_v}{4\pi r_s^2} \quad C\, m^{-2}$

(7) $\vec{E} = \vec{D}/\varepsilon_o = \hat{a}_{r_s} \frac{Q+4\pi a^2 \rho_s + \frac{4\pi}{3}(b^3 - a^3) \rho_v}{4\pi \varepsilon_o r_s^2} \quad N\, C^{-1}$

Divergence

The divergence of $\vec{D}$ equals the net flux of the vector $\vec{D}$ that flows outwardly through a closed surface S per unit volume (enclosed by $\oiint$) as the volume goes to zero.

Divergence Law

$$Div\,\vec{D} = \nabla.\vec{D} \triangleq lim_{\Delta v \to 0} \frac{\oiint \vec{D}\,.\vec{ds}}{\Delta v}$$

$$\nabla.\vec{D} = \rho_v \quad [Cm^{-3}]$$

The general form of the divergence can be written as

$$\nabla.\vec{D} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \mu_1}(h_2 h_3 D_{\mu_1}) + \frac{\partial}{\partial \mu_2}(h_1 h_3 D_{\mu_2}) + \frac{\partial}{\partial \mu_3}(h_1 h_2 D_{\mu_3})\right]$$

Where, $\mu_1, \mu_2, and\ \mu_3$ are the variables of the coordinates system, and $h_1, h_2, and\ h_3$ are the factors multiplied by the differentiable of the variables. So

For Cartesian coordinates

$$\nabla.\vec{D} = \left[\frac{\partial}{\partial x}D_x + \frac{\partial}{\partial y}D_y + \frac{\partial}{\partial z}D_z\right]$$

For Cylinderical coordinates

$$\nabla.\vec{D} = \frac{1}{r_c}\left[\frac{\partial}{\partial r_c} r_c D_{r_c} + \frac{\partial}{\partial \varphi} D_\varphi + \frac{\partial}{\partial z} r_c D_z\right]$$

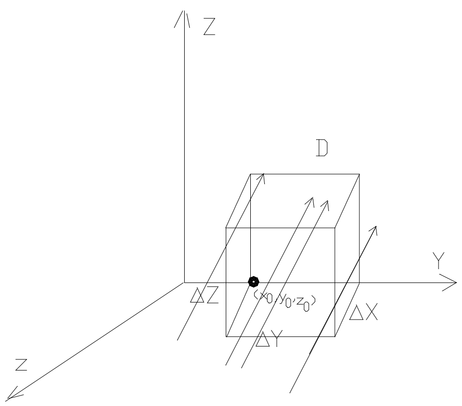
For Spherical coordinates

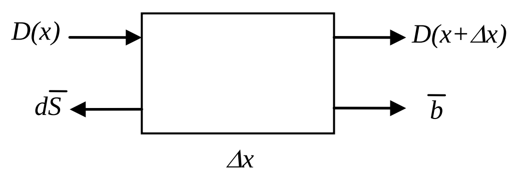
$$\nabla.\vec{D} = \frac{1}{r_s^2 \sin\theta}\left[\frac{\partial}{\partial r_s}\left(r_s^2 \sin\theta\, D_{r_s}\right) + \frac{\partial}{\partial \theta}\left(r_s \sin\theta\, D_\theta\right) + \frac{\partial}{\partial \varphi}\left(r_s D_\varphi\right)\right]$$

Proof of Divergence Law

Let a cube enclosed at its center the point (x_o,y_o,z_o) and the electric field density $\vec{D}$ crossing the cube surface at this point and is giving by:

$$\vec{D} = \hat{a}_x D_{x_o} + \hat{a}_y D_{y_o} + \hat{a}_z D_{z_o}$$





In order to express $\oiint \vec{D}\,.\,\vec{ds}$ for the cube, all six faces must be taken, the direction of $\vec{ds}$ is outward since the faces are normal to the three axes. Only one component of $\vec{D}$ will cross any two surface. Thus, It's required to

find$\oiint \overrightarrow{D} \,.\, \overrightarrow{ds}$. We take at the first the surface in + x direction and in – x direction.

$$\oint \overline{D}\cdot d\overline{S} = \left(D_{Xo}\hat{x} + D_{Yo}\hat{y} + D_{Zo}\hat{z}\right)\cdot \Delta y\Delta z\left(\hat{X}\right)\downarrow_{left} + D\left(x+\Delta x\right)\hat{x}\cdot\Delta y\Delta z\,\hat{x}\downarrow_{right}$$

$$\because D\left(x+\Delta x\right) = D\left(x_O\right) + \frac{\partial D}{\partial x}\Delta x + \cdots$$

$$\oint \overline{D}\cdot d\overline{S} = -D_{Xo}\Delta_Y\Delta z + \left[\Delta x_o * \frac{\partial D}{\partial x}\Delta x\right]\Delta y\Delta z$$

$$\oint \overline{D}\cdot d\overline{S} = \frac{\partial D}{\partial x} Dx\Delta y\Delta z$$

$$\oint \overline{D}\cdot d\overline{S}\downarrow_{backed,\,front} = \frac{\partial D_y}{\partial y} Dx\Delta y\Delta z$$

$$\oint \overline{D}\cdot d\overline{S}\downarrow_{top,bottom} = \frac{\partial D_z}{\partial z} D_x\Delta y\Delta z$$

$$\oint \overline{D}\cdot d\overline{S} = -D_{Xo}\Delta_Y\Delta z + \left[\Delta x_o * \frac{\partial D}{\partial x}\Delta x\right]\Delta y\Delta z$$

$$\oint \overline{D}\cdot d\overline{S} = \frac{\partial D}{\partial x} Dx\Delta y\Delta z$$

$$\oint \overline{D}\cdot d\overline{S}\downarrow_{backed,\,front} = \frac{\partial Dy}{\partial y} Dx\Delta y\Delta z$$

$$\oint \overline{D}\cdot d\overline{S}\downarrow_{top,bottom} = \frac{\partial Dz}{\partial z} Dx\Delta y\Delta z$$

$$\oint \overline{D}\cdot d\overline{S} = \left(\frac{\partial D_X}{\partial x} + \frac{\partial D_Y}{\partial y} + \frac{\partial D_Z}{\partial z}\right)\Delta x\Delta y\Delta z$$

$$\therefore \oint \overline{D}\cdot d\overline{S} = \left(\frac{\partial D_X}{\partial x} + \frac{\partial D_Y}{\partial y} + \frac{\partial D_Z}{\partial z}\right)\Delta v = Q$$

$$\therefore \rho_\upsilon = \frac{Q}{\Delta v} = \frac{\partial D_X}{\partial x} + \frac{\partial D_Y}{\partial y} + \frac{\partial D_Z}{\partial z} = \nabla\cdot\overline{D}$$

Example:

A charged sphere of ρ_v and radius a, the electric flux density $\overline{D}$ for $r_s < a$ is given by: $\overline{D} = \frac{10^{-5} r_s}{3} \hat{r}_s$,

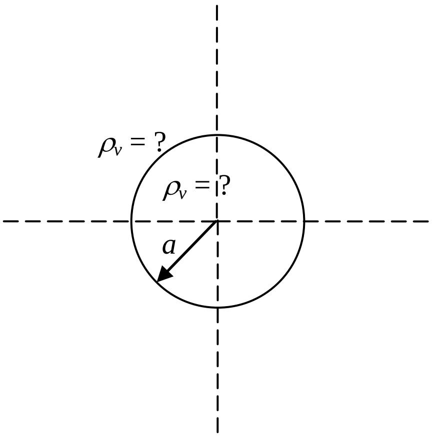
and for $r_s > a$ is given by: $\overline{D} = \frac{10^{-5} a^3}{3 r_s^{\,2}}$.

Find ρ_v in the previous two regions.

Solution:

$$\nabla \cdot \overline{D} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial r_s}\left(h_2 h_3 D_{rs}\right) + \frac{\partial}{\partial \theta}\left(h_1 h_3 D_\theta\right) + \frac{\partial}{\partial \phi}\left(h_1 h_2 D_\varphi\right)\right]$$

$$\text{where:} \quad h_1 = 1, h_2 = r_s, h_3 = r_s \sin\theta$$



for $r_s < a$:

$$\rho_v = \nabla \cdot \overline{D} = \frac{1}{r_s^{\,2} \sin\theta}\left[\frac{\partial}{\partial r_s}\left(\frac{r_s^{\,2} \sin\theta \cdot 10^{-5} r_s}{3}\right) + 0 + 0\right]$$

$$\rho_v = \nabla \cdot \overline{D} = \frac{10^{-5}}{3 r_s^{\,2}} * 3 r_s^{\,2} = 10^{-5} C/m^3$$

for $r_s > a$:

$$\rho_v = \nabla \cdot \overline{D} = \frac{1}{r_s^{\,2} \sin\theta}\left[\frac{\partial}{\partial r_s}\left(\frac{r_s^{\,2} \sin\theta \cdot 10^{-5} a^3}{3 r_s^{\,2}}\right)\right] + 0 + 0 = 0$$

Divergence Theorem:

$$\oiint \vec{D} \cdot \overrightarrow{ds} = Q_{enclosed} = \iiint \rho_v \, dv$$

From divergence law,

$$\nabla . \vec{D} = \rho_v \quad [Cm^{-3}]$$

So

$$\oiint \vec{D} \cdot \overrightarrow{ds} = Q_{enclosed} = \iiint \rho_v \, dv = \iiint \nabla . \vec{D} \, dv$$

We can transfer the surface integral into a volume integral. For the left-hand side to be equal the right hand side of divergence theorem, the following conditions must be fulfilled:

"$\vec{D}$ Must be well behaved within the volume *v* and on the surface"

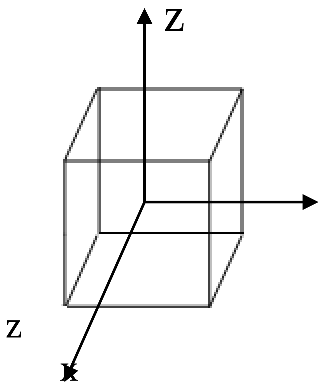
Note:

Well behaved means that $\vec{D}$ and $\nabla . \vec{D}$ are continuous and defined (not infinite).

Example:

Given $\overline{D} = \frac{10x^3}{3}\hat{x}$ evaluate both sides of the divergence theorem for the volume of cube 2m on edge centered at the origin and with edges parallel to the axis.

Solution:



$$\oint \overline{D} \cdot d\overline{S} = \int \nabla \cdot \overline{D}\, dv$$

$$L.H.S = \oint \overline{D} \cdot d\overline{S}$$

$$= \int_{-1}^{1}\int_{-1}^{1} \frac{10x^3}{3}\hat{X} \cdot dydz\hat{X} \downarrow_{X=!} + \int_{-1}^{1}\int_{-1}^{1} \frac{10x^3}{3}\hat{X} \cdot dydz\hat{X} \downarrow_{X=-!} +0+0$$

$$= \frac{10(1)^3}{3} \cdot 2 \cdot 2 = \frac{40}{3}C$$

$$= \frac{40}{3} + \frac{40}{3} = \frac{80}{3}C$$

$$R.H.S = \int \nabla \cdot \overline{D}\, dv$$

$$\therefore \nabla \cdot \overline{D} = \frac{\partial}{\partial x}\left[\frac{10x^3}{3}\right] = 10x^2$$

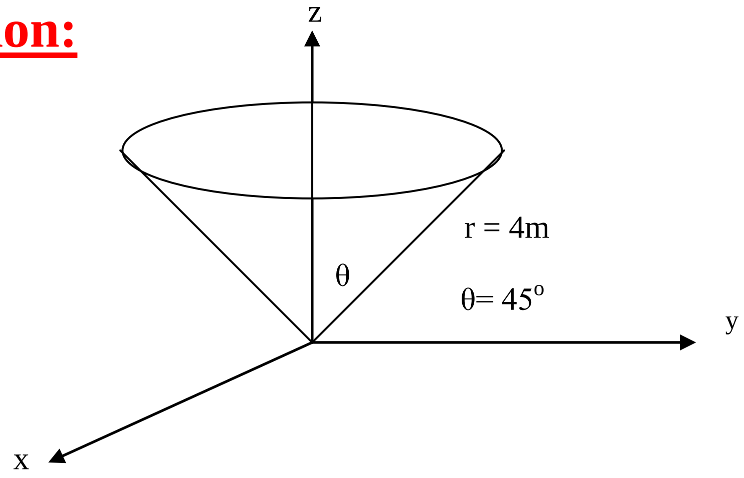
$$\therefore \int \nabla \cdot \overline{D}\, dv = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 10x^2 dxdydz = 10\left[\frac{x^3}{3}\right]_{-1}^{1} \cdot 2 \cdot 2$$

$$= \frac{10}{3}[1+1] \cdot 2 \cdot 2 = \frac{80}{3}C$$

Example:

Given $\overline{D} = \frac{5r_s^{\ 2}}{4}\hat{r}_s$ **evaluate both sides of divergence theorem for volume:** $r = 4m\ ,\ \theta = \frac{\pi}{4}$

Solution:



$$\oint \overline{D} \cdot d\overline{S} = \int \nabla \cdot \overline{D}\, dv$$

$$L.H.S = \oint \overline{D} \cdot d\overline{S}$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{5r_s^{\ 2}}{4}\hat{r}_s \cdot \hat{r}_s r_s^{\ 2} \sin\theta d\theta d\phi \downarrow_{rs=4}$$

$$= \frac{5(4)^4}{4} \cdot \int_0^{\frac{\pi}{4}} \sin\theta d\theta \cdot \int_0^{2\pi} d\phi = \frac{5(4)^4}{4}(-\cos\theta)_0^{\frac{\pi}{4}}$$

$$= 589.1\, C$$

$$\nabla \cdot \overline{D} = \frac{1}{r_s{}^2 \sin\theta} \frac{\partial}{\partial r_s}\left[r_s{}^2 \sin\theta \frac{4r_s{}^2}{4} \right]$$

$$= \frac{5}{4r_s{}^2} \cdot 4r_s{}^2 = 5r_s$$

$$RHS = \int \nabla \cdot \overline{D}\, dv = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4} 5r_s \cdot r^2 \sin\theta dr_s d\theta d\phi$$

$$= \frac{5r_s{}^4}{4}(-\cos\theta)_0^{\frac{\pi}{4}} \cdot 2\pi = 589.1C$$