CHAPTER (6)

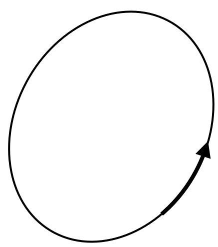
Biot-Savart law
Ampere's Circuital Law
Magnetic Field Density
Magnetic Flux

Sources of magnetic field:

1- Permanent magnet
2- Flow of current in conductors
3-Time varying of electric field inducing magnetic field

Current configurations:

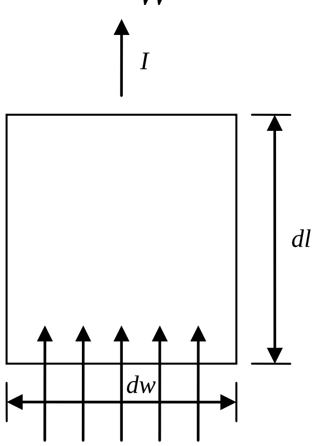
1- Filamentary current



2- Surface current: $\bar{J}_s\ (A/m)$

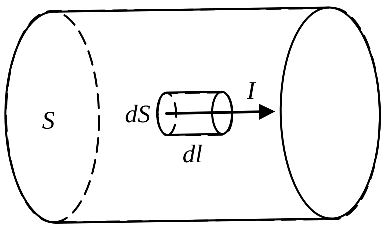
Current element: $\bar{J}_s ds\ (A.m)$,

where: $J_s = \frac{I}{W}$ ***A/m***



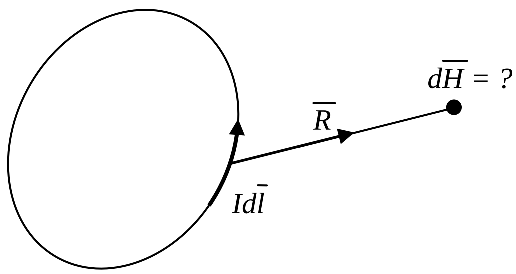
3- Volume current: $\bar{J} = \frac{I}{S}\,\hat{a}_n \; A/m^2$

Current element $\bar{J}\,dv \;\; (A.m)$, $dv = d\bar{S}.d\bar{l}$



Biot-Savart law:

$$d\bar{H} = \frac{I d\bar{l} \times \hat{a}_R}{4\pi R^2}$$



Where $\bar{H}$ = magnetic field intensity

$\hat{a}_R$ = unit vector from the current element to the point where we want to find $\bar{H}$ at it

$\bar{R}$ = distance between the current element and the point (p)

Similarly:

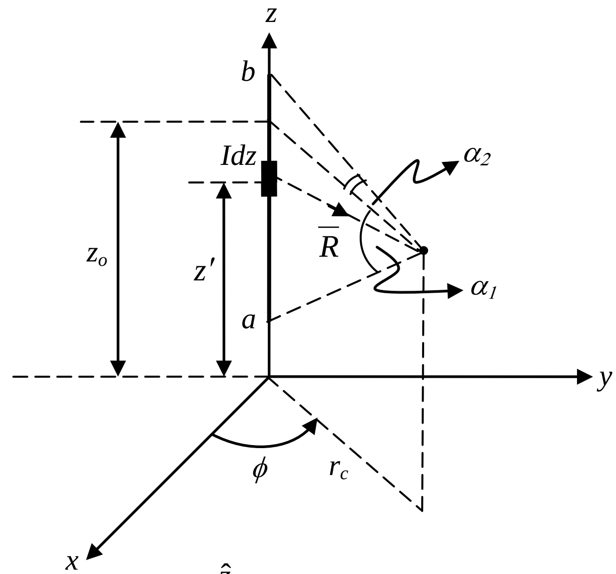
$$d\bar{H} = \frac{\bar{J}_s.ds \times \hat{a}_R}{4\pi R^2} \;\; (A/m) \;\; \rightarrow \;\; \text{Surface current element}$$

$$d\bar{H} = \frac{\bar{J}\,dv \times \hat{a}_R}{4\pi R^2} \;\; (A/m) \;\; \rightarrow \;\; \text{Volume current element}$$

The magnetic field intensity at a point P due to a wire of finite length and a current *I* passing through it along the z - axis

Derivation

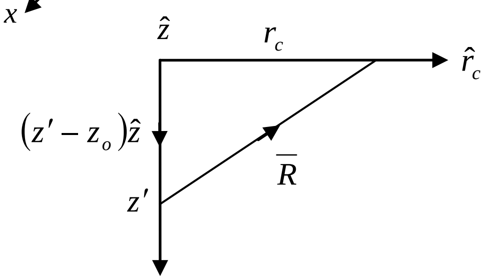
$$d\overline{H} = \frac{Id\bar{l} \times \hat{a}_R}{4\pi R^2}$$



$$Id\bar{l} = Idz'(\hat{z})$$

$$\overline{R} = r_C\,\hat{r}_C + (z_o - z')\hat{z}$$

$$R = \sqrt{r_c^2 + (z_o - z')^2}$$



$$\hat{a}_R = \frac{\overline{R}}{R} = \frac{r_C\,\hat{r}_C + (z_o - z')\hat{z}}{\sqrt{r_c^2 + (z_o - z')^2}}$$

$$\therefore\ d\overline{H} = \frac{Idz'(\hat{z}) \times [\,r_C\,\hat{r}_C + (z_o - z')\hat{z}\,]}{4\pi\left[r_c^2 + (z_o - z')^2\right]^{\frac{3}{2}}}$$

$$\therefore\ d\overline{H} = \frac{Ir_c\,dz'\,\hat{\phi}}{4\pi\left[r_c^2 + (z_o - z')^2\right]^{\frac{3}{2}}}$$

$$\overline{H} = \int d\overline{H} = \frac{Ir_c}{4\pi}\int_a^b \frac{dz'}{\left[r_c^2 + (z_o - z')^2\right]^{\frac{3}{2}}}\,\hat{\phi}$$

$$\overline{H} = \int d\overline{H} = -\frac{Ir_c}{4\pi}\int_a^b \frac{d(z_o - z')}{\left[r_c^2 + (z_o - z')^2\right]^{\frac{3}{2}}}\hat{\phi}$$

$$\because \int \frac{dx}{\left(c^2 + x^2\right)^{\frac{3}{2}}} = \frac{x}{c^2\left(c^2 + x^2\right)^{\frac{1}{2}}}$$

$$\therefore \overline{H} = \frac{-Ir_c}{4\pi}\cdot\left.\frac{(z_o - z')}{\left\{r_c^2\left[r_c^2 + (z_o - z')^2\right]\right\}^{\frac{1}{2}}}\right|_{z_a}^{z_b}\hat{\phi}$$

$$\overline{H} = \frac{-I}{4\pi r_c}\left[\frac{z_0 - z_b}{\left[r_c^2 + (z_0 - z_b)^2\right]^{\frac{1}{2}}} - \frac{z_0 - z_a}{\left[r_c^2 + (z_0 - z_a)^2\right]^{\frac{1}{2}}}\right].\hat{\phi}$$

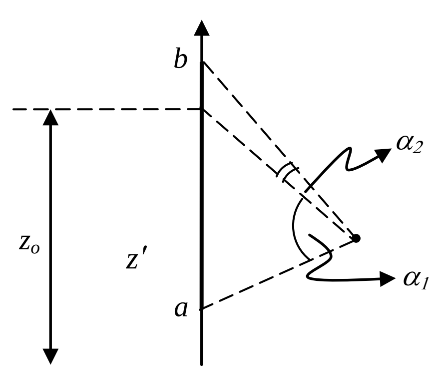
$$\overline{H} = \frac{I}{4\pi r_c}\left[\frac{z_b - z_0}{\left[r_c^2 + (z_b - z_0)^2\right]^{\frac{1}{2}}} + \frac{z_0 - z_a}{\left[r_c^2 + (z_0 - z_a)^2\right]^{\frac{1}{2}}}\right].\hat{\phi}$$

$$\overline{H} = \frac{I}{4\pi r_c}\left[\sin\alpha_2 + \sin\alpha_1\right]\hat{\phi}$$

Note:

For infinite line: $\alpha_1 = \alpha_2 = \frac{\pi}{2}$

$$\therefore \overline{H} = \frac{I}{2\pi r_c}\hat{\phi}$$



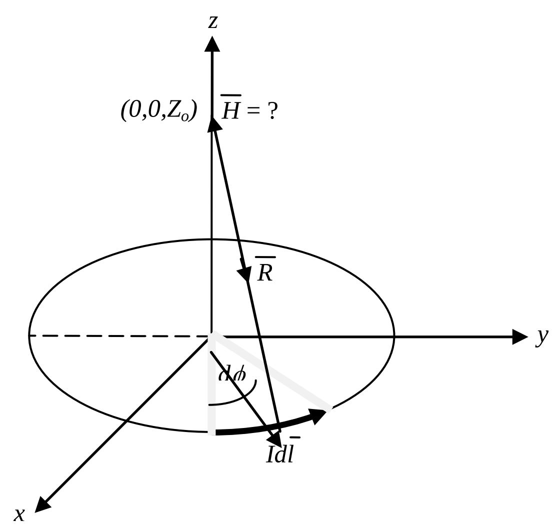
Example:

Find $\overline{H}$ at the center of circular loop carrying current *I*.

Solution:

$$d\overline{H} = \frac{Id\overline{l} \times \hat{a}_R}{4\pi R^2}$$

$$Id\overline{l} = Ia\, d\phi\, \hat{\phi}$$



$$\overline{R} = -a\hat{r}_c + z_0\hat{z}$$

$$R = \sqrt{a^2 + z_o^2}$$

$$\hat{a}_R = \frac{\overline{R}}{R} = \frac{-a\hat{r}_c + z_0\hat{z}}{\sqrt{a^2 + z_0^2}}$$

$$\therefore d\overline{H} = \frac{Ia\, d\phi\, \hat{\phi} \times (-a\hat{r}_c + z_0\hat{z})}{4\pi(a^2 + z_0^2)^{\frac{3}{2}}}$$

$$\therefore d\overline{H} = \frac{Ia^2 d\phi\, \hat{z} + Ia \quad z_0 d\phi\, \hat{r}_c}{4\pi(a^2 + z_0^2)^{\frac{3}{2}}}$$

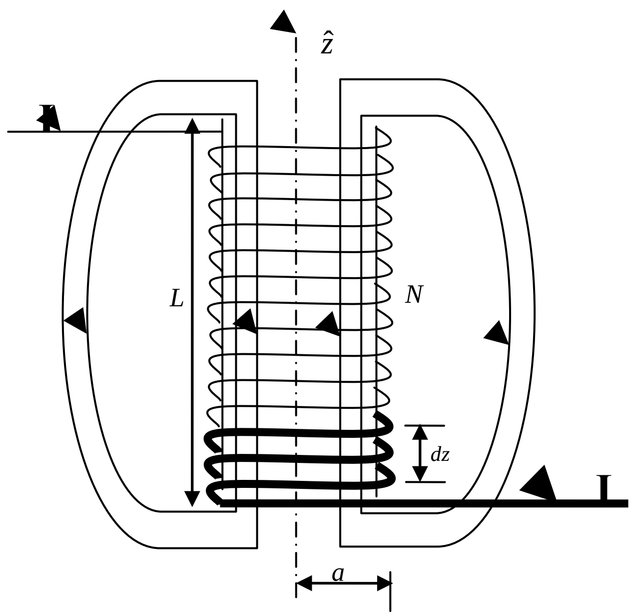
$$\overline{H} = \int d\overline{H} = \frac{Ia^2}{4\pi(a^2 + z_0^2)^{\frac{3}{2}}} \int_0^{2\Pi} d\phi . \hat{z} + \frac{Ia\, z_0}{4\pi(a^2 + z_0^2)^{\frac{3}{2}}} \int_0^{2\Pi} d\phi\, \hat{r}_c$$

$$\overline{H} = \frac{2\pi I a^2}{4\pi\left(a^2 + z_0^2\right)^{\frac{3}{2}}}\hat{z} = \frac{Ia^2}{2\left(a^2 + z_0^2\right)^{\frac{3}{2}}}\hat{z}$$

Example:

Find the magnetic field intensity at the center of a solenoid (coil) of radius *a* and length *L* and the number of *N* turns carrying current *I*.

Solution:



Current per unit width = $J_s = \dfrac{NI}{L}$

I dl = $Ia_{d\phi}$ = J_s ds = J_s $a_{d\phi}$dz

So, Current in length *dz* is

$$I = J_s\, dz = \frac{(NI)}{l} dz$$

Since $\overline{H}$ due to circular loop of current I and radius a is given by

$$\overline{H} = \frac{Ia^2}{2\left(a^2 + z_0^2\right)^{\frac{3}{2}}}\hat{z}$$

So

$$d\overline{H} = \frac{\frac{(NI)}{l} dz.a^2}{2(a^2 + z^2)^{\frac{3}{2}}} \hat{z}$$

$\overline{H}$ **at the center of the solenoid is given by**

$$\overline{H} = \frac{NIa^2}{2l} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dz}{(a^2 + z^2)^{\frac{3}{2}}} \hat{z} = \frac{NIa^2}{2l} \left(\frac{z}{a^2 (a^2 + z^2)^{\frac{1}{2}}} \right)_{-\frac{L}{2}}^{\frac{L}{2}} \hat{z}$$

$$= \frac{2\left(\frac{L}{2}\right) NI}{2l \left[a^2 + \left(\frac{L}{2}\right)^2 \right]^{\frac{1}{2}}} \hat{z} = \frac{NI}{2\left(a^2 + \frac{L^2}{4}\right)^{\frac{1}{2}}} \hat{z} = \frac{NI}{2\left(\frac{4a^2 + L^2}{4}\right)^{\frac{1}{2}}} \hat{z}$$

$$\therefore \overline{H} = \frac{NI}{(4a^2 + L^2)^{\frac{1}{2}}} \hat{z}$$

If $L >> a$: $$\therefore \overline{H} = \frac{NI}{L} \hat{z}$$

<u>Notes:</u>

For the magnetic field at the end of the solenoid we must integrate from

0 $\rightarrow$ ***L*****, so** $\overline{H}$ **at the end:**

$$\overline{H} = \frac{NIa^2}{2L}\int_0^L \frac{dz}{(a^2+z^2)^{3/2}}\hat{z} = \frac{NIa^2}{2L} * \frac{z}{a^2(a^2+z^2)^{\frac{1}{2}}}\Bigg|_0^L \hat{z}$$

$$= \frac{NI}{2L} * \frac{l}{(a^2+z^2)^{\frac{1}{2}}}\hat{z} = \frac{NI}{2(a^2+L^2)^{\frac{1}{2}}}\hat{z}$$

For *a* << *L* : $\therefore \overline{H} = \frac{NI}{2L}\hat{z} \quad A/m$

This means that the magnetic field intensity at the end of the solenoid is approximately half its value at its center.

Ampere's circuital law:

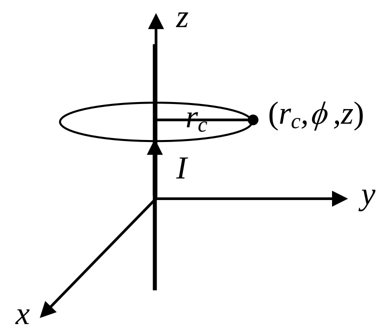
It states that the line integral of the tangential component of magnetic field intensity around a closed path is equal to the current enclosed by the path

$$\oint \overline{H}.d\overline{l} = I_{en}$$

Example:

Find the magnetic field intensity at a point (r_c, ϕ, z) due to infinite wire of current *I*.

Solution:



1- Ampere's circuital law

$$\oint \overline{H}.d\overline{l} = I_{en}$$

2- Choice of Amperian loop

3- I_{en} = I

4- $\oint \overline{H}.d\overline{l}$ = $H.2\pi r_c$

5- $H.2\pi r_c = I$

6- $\therefore \overline{H} = \dfrac{I}{2\pi r_c}\hat{\phi}$ A/m

Conditions for application of Ampere's law:

$\overline{H}$ must be constant on the loop.

Example:

Find $\overline{H}$ inside and outside a conductor (magnetic material) of infinite length carrying current *I* and of radius *a*.

Solution:

Region (I) $r_c < a$

1- Ampere's circuital law

$$\oint \overline{H}.d\overline{l} = I_{en}$$

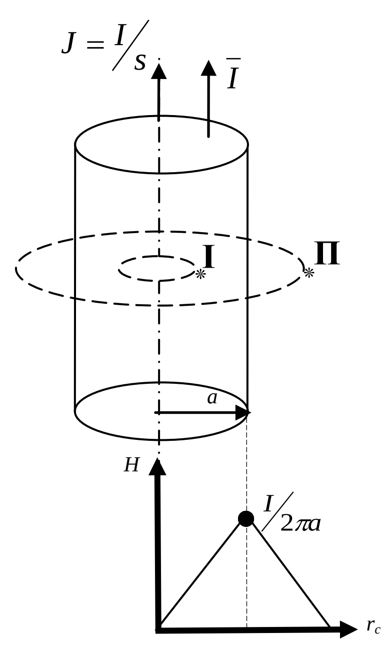
2- Choice of Amperian loop

3- $I_{en} = J.S = \dfrac{I}{\pi a^2}.\pi r_c^2$

4- $\oint \overline{H}.d\overline{l} = H.2\pi r_c$

5- $H.2\pi r_c = \dfrac{I}{\pi a^2}.\pi r_c^2$

6- $\overline{H} = \dfrac{I r_c}{2\pi a^2}\hat{\phi} \rightarrow$ (1)



Region (II) $r_c > a$

1-Ampere's circuital law

$$\oint \overline{H}.d\overline{l} = I_{en}$$

2-Choice of Amperian loop

3- $I_{en} = J.S = I$

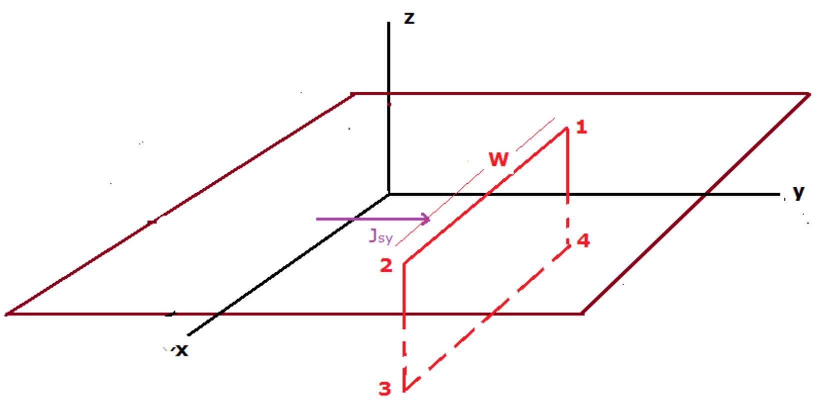
4- $\oint \overline{H}.d\overline{l} = H.2\pi r_c$

5- $H.2\pi r_C = I$

6- $\overline{H} = \frac{I}{2\pi r_c}\hat{\phi} \rightarrow (2)$

The magnetic field intensity above and below a surface current distribution of infinite extended sheet with surface current density J_s

Derivation



1-Ampere's circuital law

$$\oint \overline{H}.d\overline{l} = I_{en}$$

2-Choice of Amperian loop (1-2-3-4) (x-z plane)

3- $I_{en} = J_{sy} * W$

4- $\oint \overline{H}.\overline{dl} = \int_1^2 \overline{H}.d\bar{l} + \int_2^3 \overline{H}.d\bar{l} + \int_3^4 \overline{H}.d\bar{l} + \int_4^1 \overline{H}.d\bar{l}$

above the surface :

$$d\overline{H} = \frac{Id\overline{l} \times \hat{a}_R}{4\pi R^2} = (\ )\hat{y} \times \hat{z} = \hat{x}$$

below :

$$d\overline{H} = (\quad)\hat{y} \times (-\hat{z}) = -(\ )\hat{x}$$

$$\oint \overline{H}.\overline{dl} = \int_1^2 H_x \hat{x}.dx\,\hat{x} + 0 + \int_3^4 H_x(-\hat{x}).dx(-\hat{x}) + 0$$

5- $H_x W + H_x W = J_{sy} W$

6- $H_x = \dfrac{J_{sy}}{2}$

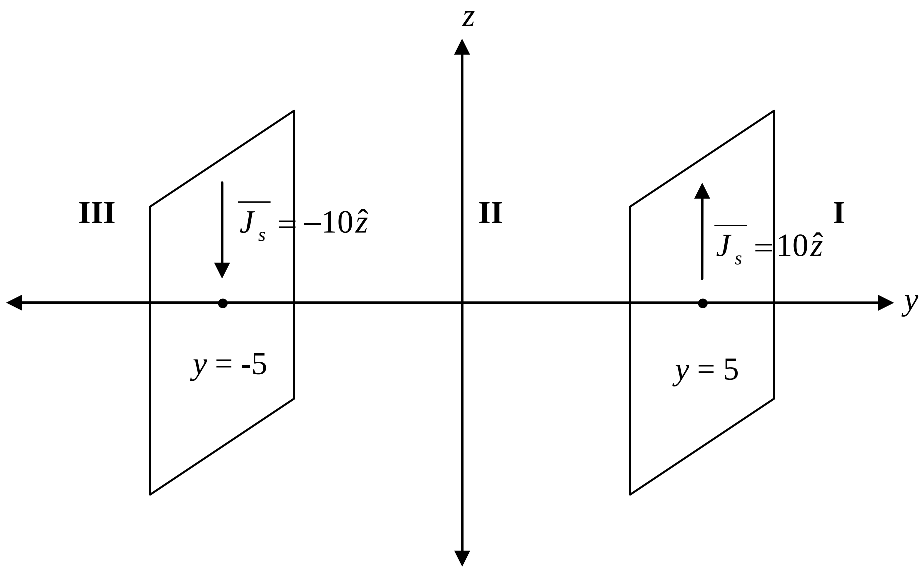
Generally : $\overline{H} = \dfrac{1}{2}\overline{J_s} \times \hat{n}$

where $\overline{J_s} \equiv$ surface current denisty

$\hat{n} \equiv$ unit vector $\perp$ to the surface

Example:

Find $\overline{H}$ in all regions for the following current configuration shown in figure.



Solution:

Region (I): $y > 5$

$$\overline{H}_1 = \frac{\overline{J}_s}{2} \times \hat{n} = \frac{10\hat{z}}{2} \times \hat{y} = \frac{-10}{2}\hat{x}$$

$$\overline{H}_2 = \frac{-10}{2}\hat{z} \times \hat{y} = \frac{10}{2}\hat{x}$$

$$\overline{H}_t = \overline{H}_1 + \overline{H}_2 = 0$$

Region (II): $-5 < y < 5$

$$\overline{H}_1 = \frac{1}{2}\overline{J}_{s1} \times \hat{n} = \frac{1}{2} 10\,\hat{z} \times (-\hat{y}) = 5\hat{x}$$

$$\overline{H}_2 = \frac{1}{2}\overline{J}_{s2} \times \hat{n} = \frac{1}{2}(-10\hat{z}) \times (\hat{y}) = 5\hat{x}$$

$$\overline{H}_t = 10\,\hat{x}$$

$$\text{Region (III)}: \quad y < 5$$

$$\overline{H_1} = \frac{1}{2}(10\hat{z}) \times (\hat{y}) = 5\hat{x}$$

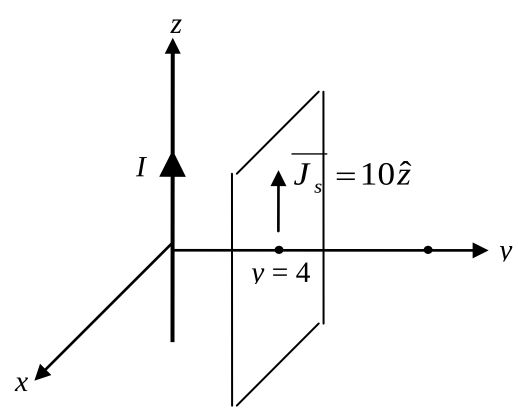
$$\overline{H_2} = \frac{1}{2}(-10\hat{z}) \times (\hat{y}) = -5\hat{x}$$

$$\overline{H_t} = \text{zero}$$

Example:

Find $\overline{H}$ in all regions for the following current configuration shown in figure.

Solution:



$$\text{Region (I)}: \quad y > 4$$

$$\overline{H_s} = \frac{1}{2}\overline{J_S} \times \hat{n} = \frac{1}{2}(10\hat{z}) \times \hat{y} = -5\hat{x}$$

$$\oint \overline{H}.\overline{dl} = I_{en}$$

$$H.2\pi r_c = I$$

$$\overline{H} = \frac{I}{2\pi . r_c}\hat{\phi}$$

$$d\overline{H} = \frac{Id\overline{l} \times \hat{a}_R}{4\pi R^2} = \qquad | \qquad | \; \hat{\phi}$$

$$\hat{\phi} = \hat{z} * \hat{y} = -\hat{x}$$

$$\overline{H_l} = \frac{I}{2\pi . r_c}(-\hat{x})$$

$$\therefore \overline{H_t} = \overline{H_s} + \overline{H_l} = -5\hat{x} - \frac{I}{2\pi . r_c}\hat{x} \quad A/m$$

Region (II): $0 < y < 4$

$$\overline{H_s} = \frac{1}{2}\overline{J_S} \times \hat{n} = \frac{1}{2}(10\hat{z}) \times (-\hat{y}) = 5\hat{x}$$

$$\overline{H_l} = \frac{I}{2\pi . r_c}\hat{\phi}$$

$$\hat{\phi} = \hat{z} * \hat{y} = -\hat{x}$$

$$\overline{H_l} = \frac{I}{2\pi . r_c}(-\hat{x})$$

$$\overline{H_t} = \overline{H_s} + \overline{H_l} = 5\hat{x} - \frac{I}{2\pi . r_c}\hat{x} \quad A/m$$

Region (III): $y < 0$

$$\overline{H_s} = \frac{1}{2}\overline{J_S} \times \hat{n} = \frac{1}{2}(10\hat{z}) \times (-\hat{y}) = 5\hat{x}$$

$$\overline{H_l} = \frac{I}{2\pi . r_c}\hat{\phi}$$

$$\hat{\phi} = \hat{z} * -\hat{y} = \hat{x}$$

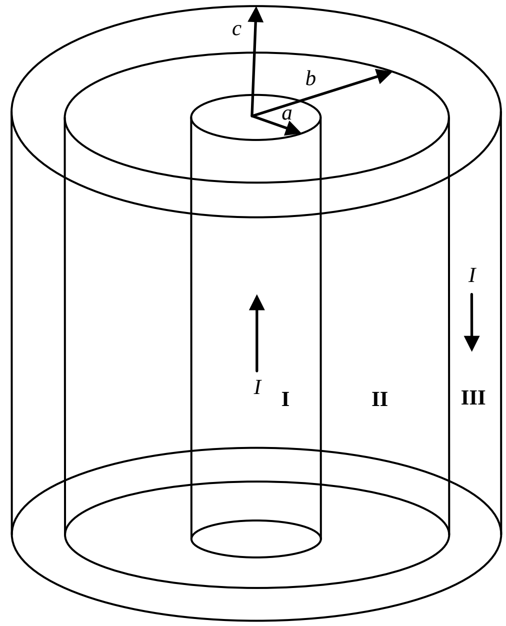
$$\overline{H_l} = \frac{I}{2\pi . r_c}(\hat{x})$$

$$\overline{H_t} = \overline{H_s} + \overline{H_l} = 5\hat{x} + \frac{I}{2\pi . r_c}\hat{x} \quad A/m$$

Example:

Find $\overline{H}$ in all region for coaxial cable of radii *a, b, c* carrying *I* in inner conductor and *I* in outer conductor

Solution:



Region (I) $r_c < a$

1-Ampere's circuital law

$$\oint \overline{H}.d\overline{l} = I_{en}$$

2-Choice of Amperian loop

3- $I_{en} = J.S = \dfrac{I}{\pi a^2}.\pi r_c^2$

4- $\oint \overline{H}.d\overline{l} = H.2\pi r_c$

5- $H.2\pi r_c = \dfrac{I}{\pi a^2}.\pi r_c^2$

6- $\overline{H} = \dfrac{I r_c}{2\pi a^2}\hat{\phi} \quad \rightarrow \quad (1)$

Region (II) $a < r_c < b$

1-Ampere's circuital law

$$\oint \overline{H}.d\overline{l} = I_{en}$$

2-Choice of Amperian loop

3- $I_{en} = J.S = I$

4- $\oint \overline{H}.d\overline{l} = H.2\pi r_c$

5- $H.2\pi r_C = I$

6- $\overline{H} = \dfrac{I}{2\pi r_c}\hat{\phi} \rightarrow (2)$

Region (III) $b < r_c < c$

1-Ampere's circuital law

$$\oint \overline{H}.d\overline{l} = I_{en}$$

2-Choice of Amperian loop

3- $I_{en} = I - J.S$

$$= I - \left[\frac{I}{\pi\left(c^2 - b^2\right)} \quad \pi\left(r_c^2 - b^2\right)\right]$$

$$= I\left(1 - \frac{r_c^2 - b^2}{c^2 - b^2}\right)$$

4- $\oint \overline{H}.d\overline{l} = H.2\pi r_c$

5- $H.2\pi\, r_C = I\left(1 - \frac{r_c^2 - b^2}{c^2 - b^2}\right)$

6- $\overline{H} = \frac{I}{2\pi r_c}\left(1 - \frac{r_c^2 - b^2}{c^2 - b^2}\right)\hat{\phi}$ **(3)**

Region (IV) $r_c > c$

1-Ampere's circuital law

$$\oint \overline{H}.d\overline{l} = I_{en}$$

2-Choice of Amperian loop

3- I_{en} **= I –I =0**

4- $\oint \overline{H}.d\overline{l}$ **=** $H.2\pi r_c$

5- $H.2\pi r_c = 0$

6- $\overline{H} = 0$

Curl law

$$\overline{J} = \nabla \times \overline{H} \qquad \text{A.m}^{-2}$$

Stock's theorem:

$$\text{but}: \oint \overline{H}.d\overline{l} = I_{en} = \int \overline{J}.d\overline{S}$$

$$\therefore \oint \overline{H}.d\overline{l} = \int \nabla \times \overline{H}.d\overline{S} = \int \overline{J}.d\overline{S}$$

Example:

$\overline{H} = \dfrac{Ir_c}{2\pi a^2}\hat{\phi}$. Find $\overline{J}$.

Solution:

$$\nabla \times \overline{H} = \frac{1}{h_1 h_2 h_3}\begin{bmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 H_{u1} & h_2 H_{u2} & h_3 H_{u3} \end{bmatrix}$$

$$\overline{J} = \nabla \times \overline{H} = \frac{1}{r_c}\begin{bmatrix} \hat{r}_c & r_c\,\hat{\phi} & \hat{z} \\ \dfrac{\partial}{\partial r_c} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{r_c^2\, I}{2\pi\, a^2} & 0 \end{bmatrix}$$

$$= \frac{1}{r_c}\left[0\,\hat{r}_c + 0\,\hat{\phi} + \frac{\partial}{\partial r_c}\left(\frac{r_c^2 I}{2\pi a^2}\right)\hat{z}\right] = \frac{2r_c I}{2r_c \pi a^2}\,\hat{z} = \frac{I}{\pi a^2}\,\hat{z}$$

Magnetic flux density $\overline{B}$

$$\overline{B} = \mu_0 \mu_r \overline{H}$$

Magnetic flux ϕ_m:

$$\phi_m = \iint \overline{B}.d\overline{S}$$

For a closed surface:

$$\phi_m = \oiint \overline{B}.d\overline{S} = 0$$

Applying divergence theorem

$$\oiint \overline{B}.d\overline{S} = \iiint \nabla.\overline{B}\, dv = 0$$

$$\therefore\ \nabla.\overline{B} = zero,\ \text{But}\ \oint \overline{E}.d\overline{l} = zero$$

Due to the conservative property of the field. Applying Stocke's theorem

$$\oint \overline{E}.d\overline{l} = -\int_s\int \nabla\times\overline{E}.d\overline{S} = 0$$

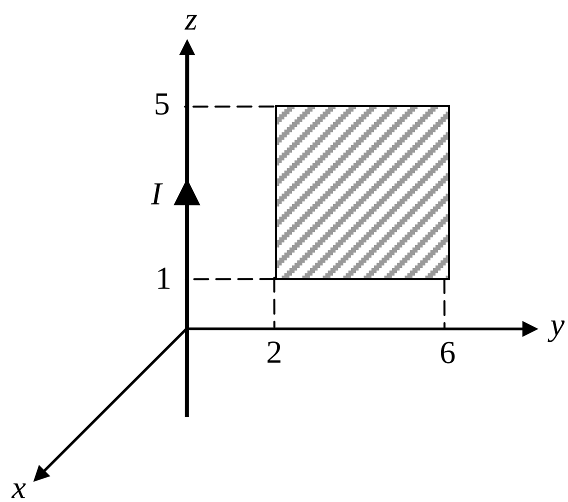
$$\therefore \nabla\times\overline{\mathrm{E}} = 0$$

Maxwell's equations in electro and magneto static fields

Differential Form	Integral Form
$\nabla\times\overline{H} = \overline{J}$	$\int \overline{H}.d\overline{l} = I$
$\nabla\times\overline{E} = 0$	$\oint \overline{E}.d\overline{l} = 0$
$\nabla.\overline{D} = \rho_v$	$\oint \overline{D}.d\overline{S} = Q_{en}$
$\nabla.\overline{B} = 0$	$\oint \overline{B}.d\overline{S} = 0$
$\nabla.\overline{J} + \frac{\partial\rho_v}{\partial t} = 0$	$\int \overline{J}.d\overline{S} = -\frac{\partial Q}{\partial t}$

Example:

Find the total magnetic flux that crossing the area shown in figure.



Solution:

$$\overline{H} = \frac{I}{2\pi r_c}\hat{\phi}$$

$$\overline{H} = \frac{I}{2\pi y}\hat{\phi}$$

$$\overline{B} = \mu_0 \overline{H} = \frac{\mu_0 I}{2\pi y}\hat{\phi}$$

$$\because d\overline{H} = \frac{I d\bar{l} \times \hat{a}_R}{4\pi R^2} = [\quad]\hat{\phi}$$

$$\therefore \hat{\phi} = \hat{z} \times \hat{y} = -\hat{x}$$

$$\therefore \overline{B} = -\frac{\mu_o I}{2\pi y}\hat{x}$$

$$d\overline{S} = dydz(-\hat{x})$$

$$\phi_m = \int_1^5 \int_2^6 \frac{\mu_o I}{2\pi y} dydz$$

$$\phi_m = \frac{\mu_o I}{2\pi} (\ln y)_2^6 (z)_1^5$$

$$\phi_m = \frac{4\mu_o I}{2\pi} [\ln 6 - \ln 2] \qquad \text{Weber}$$