CHAPTER (8)

TIME VARYING ELECTRIC AND MAGNETIC FIELDS

Faraday's law:

Due to the first experiment of Faraday, we can say that a time-varying magnetic field produces an electromotive force (emf) which may establish a current in a suitable closed circuit. Faraday's law is stated as

$$emf = -\frac{\partial \psi_m(t)}{\partial t} \qquad [V] \qquad (1)$$

A non zero value of $\left(\frac{\partial \psi_m(t)}{\partial t}\right)$ may result from any of the following situations:

- **A time-changing flux linking a stationary closed path.**
- **Relative motion between a steady flux and a closed path.**
- **A combination of the two previous situations.**

Note:

1-The minus sign is an indication that the emf is in such a direction as to produce a current whose flux, if added to the original flux, would reduce the magnitude of the emf.
This statement that the induced voltage acts to produce an opposing flux is known as Lenz's law.

2-For N – turns filamentary conductor closed path

$$emf = -N\frac{\partial \psi_m(t)}{\partial t} \qquad (2)$$

The emf is considered as the voltage about a specific closed path, and it is defined as

$$e.m.f = \oint \vec{E}\ .\overrightarrow{dl} = -\frac{\partial \psi_m}{\partial t} \qquad (3)$$

In electro-statics, equation (3) must lead to zero potential difference about a closed path. In time varying fields, the line integral leads to an emf or a potential difference.
Equation (3) can be written as

$$e.m.f = \oint \vec{E}\ .\overrightarrow{dl} = -\frac{\partial \psi_m}{\partial t} = -\frac{\partial}{\partial t}\iint \vec{B}\ .\overrightarrow{ds} \quad (4)$$

In general, Faraday's law manifests itself in either or both a stationary circuit linked by a time varying magnetic flux, such as a transformer, or the magnetic flux may be static, but the circuit is moving relative to the flux in such a way as to produce a time varying flux enclosed by the circuit. A rotating machine generates an emf by the latter mechanism.

1^{st} $Maxwell's\ equation$

Applying Stock's theorem on equation (4)

$$e.m.f = \oint \vec{E}\ .\overrightarrow{dl} = \iint \nabla x\vec{E}\ .\overrightarrow{ds} = -\frac{\partial}{\partial t}\iint \vec{B}\ .\overrightarrow{ds}$$

So, $1^{st}\ Maxwell's\ equation\ in\ differential\ form\ is$

$$\nabla x\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (5)$$

And, 1^{st} ***Maxwell's equation in integral form is***

$$\oint \vec{E} \cdot \overrightarrow{dl} = \iint \nabla x \vec{E} \cdot \overrightarrow{ds} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot \overrightarrow{ds} \qquad (6)$$

Displacement current density $\vec{J}_d$

It is shown from 1^{st} ***Maxwell'sequation*** in differential form (5)

$$\nabla x \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

that a time varying magnetic field produces an electric field $\vec{E}$. From the definition of the curl, the electric field $\vec{E}$ has the special property of circulation; its line integral about a general closed path is not zero.

Now we will consider the time varying electric field. From Curl law as it applies to steady magnetic fields, where

$$\nabla \, x \, \vec{H} = \vec{J}_c \qquad (7)$$

Taking the divergence of both sides of equation (7),

$$\nabla \cdot \nabla \, x \, \vec{H} = \nabla \cdot \vec{J}_c \qquad (8)$$

Mathematically, the left hand side of equation (8) equals zero ($\nabla . \nabla \, x \, \vec{H} = 0$), while the right hand side of equation (8), from continuity equation (conservation of charge) equals to

$$\nabla \cdot \vec{J}_c = -\frac{\partial \rho_v}{\partial t} \qquad (9)$$

So, equation (8) becomes

$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot \vec{J}_c = -\frac{\partial \rho_v}{\partial t} \qquad (10)$$

Since in equation (10), the L.H.S = 0, but the R.H.S ≠ 0, so there is a missing term that appear in time varying field as that appear in first Maxwell's equation $\left(-\frac{\partial \vec{B}}{\partial t}\right)$, in order to fulfill that R.H.S = L.H.S. This missing term that must be added to the L.H.S is $\left(+\frac{\partial \rho_v}{\partial t}\right)$. So

$$\nabla \cdot \nabla \times \vec{H} = -\frac{\partial \rho_v}{\partial t} + \frac{\partial \rho_v}{\partial t} \qquad (11)$$

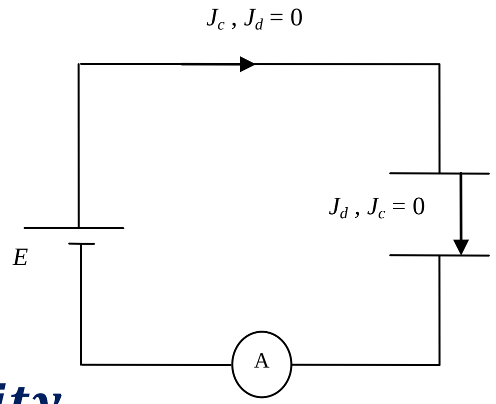
Using the divergence law, $\nabla \cdot \vec{D} = \rho_v$, equation (11) becomes

$$\nabla \cdot \nabla \times \vec{H} = -\frac{\partial \rho_v}{\partial t} + \frac{\partial}{\partial t}(\nabla \cdot \vec{D})$$

$$\nabla \cdot \left[\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t}\right] = -\frac{\partial \rho_v}{\partial t} = \nabla \cdot \vec{J}_c$$

So,

$$\nabla \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$$



$$\nabla \times \vec{H} = \vec{J}_c + \vec{J}_d$$

$\vec{J}_c$ is the coduction current density

$\vec{J}_d$ is the displacement current density

So, 2^{nd} *Maxwell's equation in differential form is*

$$\nabla \times \vec{H} = \vec{J}_c + \vec{J}_d \qquad (12)$$

And, 2^{nd} *Maxwell's equation in integral form is*

$$\oint \vec{H} \cdot \vec{dl} = \iint \nabla \times \vec{H} \cdot \vec{ds} = \iint \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds} + \iint \vec{J}_c \cdot \vec{ds} \quad (13)$$

3^{rd} Maxwell's equation:

$$\oint \overline{B} \cdot d\overline{S} = 0 \quad \rightarrow \quad 3^{rd} \text{ Maxwell's equation in Integral form}$$

Applying the divergence theorem

$$\oint \vec{B} \cdot \vec{ds} = \iiint \nabla \cdot \vec{B} \, dv = 0$$

So, 3^{rd} *Maxwell's equation in differential form is*

$$\nabla \cdot \vec{B} = 0 \qquad (14)$$

And, 3^{rd} *Maxwell's equation in integral form is*

$$\oint \vec{B} \cdot \vec{ds} = \iiint \nabla \cdot \vec{B} \, dv = 0 \qquad (15)$$

4^{th} Maxwell's equation:

Applying the divergence theorem on the electric flux density $\vec{D}$, we have

$$\oint \vec{D} \cdot \overrightarrow{ds} = Q_{en} = \iiint \nabla \cdot \vec{D}\, dv = \iiint \rho_v\, dv$$

So, 4^{th} *Maxwell's equation in differential form is*

$$\nabla \cdot \vec{D} = \rho_v \qquad (16)$$

And, 4^{th} *Maxwell's equation in integral form is*

$$\oint \vec{D} \cdot \overrightarrow{ds} = Q_{en} = \iiint \nabla \cdot \vec{D}\, dv = \iiint \rho_v\, dv \qquad (17)$$

Summary:

Maxwell's equation	
Differential form	**Integral form**
$\nabla x \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint \vec{E} \cdot \overrightarrow{dl} = \iint \nabla x \vec{E} \cdot \overrightarrow{ds} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot \overrightarrow{ds}$
$\nabla\, x\, \vec{H} = \vec{J}_c + \vec{J}_d$	$\oint \vec{H} \cdot \overrightarrow{dl} = \iint \nabla\, x\, \vec{H} \cdot \overrightarrow{ds} = \iint \frac{\partial \vec{D}}{\partial t} \cdot \overrightarrow{ds} + \iint \vec{J}_c \cdot \overrightarrow{ds}$
$\nabla \cdot \vec{D} = \rho_v$	$\oint \vec{D} \cdot \overrightarrow{ds} = Q_{en} = \iiint \nabla \cdot \vec{D}\, dv = \iiint \rho_v\, dv$
$\nabla \cdot \vec{B} = 0$	$\oint \vec{B} \cdot \overrightarrow{ds} = \iiint \nabla \cdot \vec{B}\, dv = 0$

$$\vec{J}_d = \frac{\partial \overline{D}}{\partial t} = \frac{\varepsilon_0 \partial \overline{E}}{\partial t}$$

$$\nabla \cdot \overline{J}_c + \frac{\partial \rho_v}{\partial t} = 0$$

$$I_d = \vec{J}_d \cdot \vec{A}$$

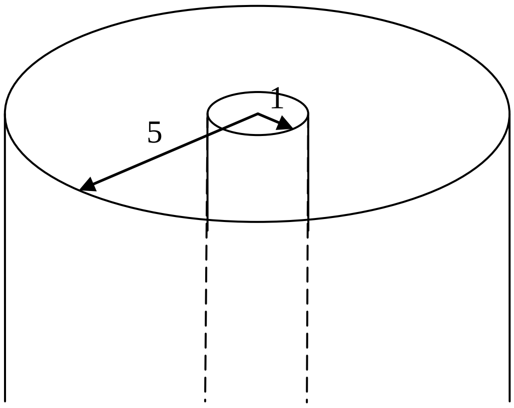
Example:

A coaxial cable of $\varepsilon_r = 4$, the inner conductor radius = 1mm and the outer conductor radius = 5mm. Find the displacement current between the two conductors per meter if the voltage difference equal 100 cos $(12\pi.10^6 t)$ volt.

Solution:

1- Using circuit concept

For a coaxial cable:



$$C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)} = \frac{2\pi * 8.85 * 10^{-12} * 4}{\ln\left(\frac{5}{1}\right)}$$

$$I_d = C\frac{dV}{dt}$$

$$\frac{dV}{dt} = 100\sin(12\pi * 10^6 t) * 12\pi * 10^6$$

$$I_d = \frac{2\pi * 8.85 * 10^{-12} * 4}{\ln\left(\frac{5}{1}\right)} 100\sin(12\pi * 10^6 t) * 12\pi * 10^6 \text{ A/m}$$

2- Using electro-magnetic field concept

The displacement current density $\vec{J}_d$ can be determined from $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$, after determining $\vec{D}$ by applying the Gauss's law as follows:

1) $\oint \vec{D} \cdot \vec{ds} = Q_{en}$

2) *choice of Gaussian surface*

3) $Q_{en} = \rho_l \, l$

4) $\oint \vec{D} \cdot \vec{ds} = 2\pi r_c l \, D$

5) $2\pi r_c l \, D = \rho_l \, l$

6) $\vec{D} = \hat{a}_{r_c} \frac{\rho_l}{2\pi r_c} \, C/_{m^2}$

7) $\vec{E} = \hat{a}_{r_c} \frac{\rho_l}{2\pi\varepsilon_o\varepsilon_r r_c} \, N/_C$

In order to determine ρ_l in terms of the potential difference V, the potential V is determined as

$$V_{ab} = -\int_b^a \vec{E}\,.\,\vec{dl}$$

$$V_{ab} = -\int_5^1 \hat{a}_{r_c}\,\frac{\rho_l}{2\pi\varepsilon_o\varepsilon_r r_c}\,.\,\hat{a}_{r_c} dr_c$$

$$100\cos(12\pi.10^6 t) = \frac{\rho_l}{2\pi\varepsilon_o\varepsilon_r}\,ln\,(5)$$

$$\rho_l = \frac{200\,\pi\varepsilon_o\varepsilon_r\cos(12\pi.10^6 t)}{ln\,(5)}$$

So,

$$\vec{D} = \hat{a}_{r_c}\frac{\rho_l}{2\pi r_c} = \hat{a}_{r_c}\frac{200\,\pi\varepsilon_o\varepsilon_r\,cos\,(12\pi.10^6 t)}{2\pi r_c ln\,(5)}$$

$$\vec{J}_d = \frac{\partial\vec{D}}{\partial t} = -\hat{a}_{r_c}\frac{200\,\pi\varepsilon_o\varepsilon_r\,(12\pi.10^6)sin\,(12\pi.10^6 t)}{2\pi r_c ln\,(5)}$$

$$I_d = \iint \vec{J}_d\,.\,\vec{ds}$$

$$I_d = \int_0^{2\pi}\int_0^{l} -\hat{a}_{r_c}\frac{200\,\pi\varepsilon_o\varepsilon_r\,(12\pi.10^6)sin\,(12\pi.10^6 t)}{2\pi r_c ln\,(5)}\,.\,\hat{a}_{r_c} r_c d\varphi dz$$

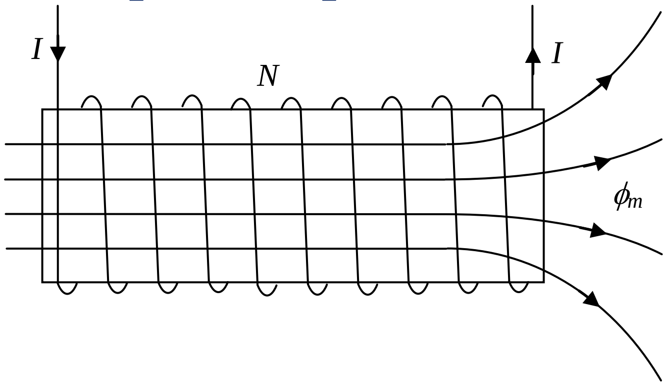
$$I_d = \frac{200\,\pi\varepsilon_o\varepsilon_r\,(12\pi.10^6)sin\,(12\pi.10^6 t)}{ln\,(5)}\,l$$

$$\frac{I_d}{l} = \frac{200\,\pi\varepsilon_o\varepsilon_r\,(12\pi.10^6)sin\,(12\pi.10^6 t)}{ln\,(5)}\,A/m$$

Inductance:

When dealing with time – varying current and thus time – varying magnetic fields, the inductance of the inductor as a circuit concept becomes quite important and is defined as

$$L = \frac{\Lambda}{I} \quad [H]$$



Where:
Λ (lambda) is the total flux linkage of the inductors and I is the current flowing in the inductor.

A flux linkage of one exists when 1 Wb of flux links one turn of the inductor.
So, if all the flux link all the turns, the total flux linkage is equal to

$$\Lambda = N\,\psi_m \qquad [\text{Wb turns}]$$

Where, ψ_m is the total flux produced by the inductor

L is the self inductance
If the coil has N turns:

$$L = \frac{\Lambda}{I} = \frac{N\,\psi_m}{I}$$

Example:

Find the self inductance of the solenoid shown in figure.

Solution:

$$L = \frac{\Lambda}{I} = \frac{N\,\psi_m}{I}$$

$$\psi_m = \iint \vec{B}\,.\,\vec{ds} = B\,A = B\,(\pi a^2)$$

$$\vec{B} = \mu_o \mu_r \vec{H}$$

The magnetic field intensity $\vec{H}$ can be determined by applying Ampere's circuital law as follows:

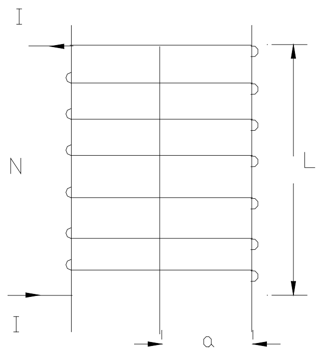
(1) $\oint \vec{H}\,.\,\vec{dl} = I_{en}$

(2) *Choice of Amperian loop*

(3) $I_{en} = N\,I$

(4) $\oint \vec{H}\,.\,\vec{dl} = H\,l$

(5) $\vec{H} = \frac{N\,I}{l}\,\hat{a}_z$



So,

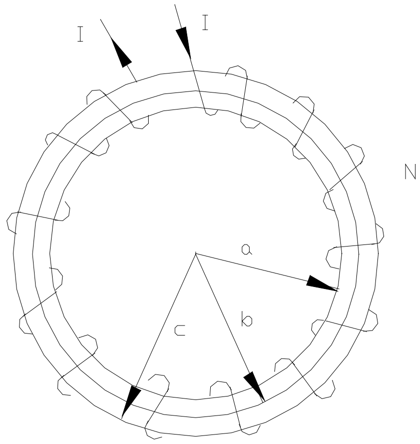
$$\vec{B} = \mu_o \mu_r \vec{H} = \mu_o \mu_r \frac{N\,I}{l}\,\hat{a}_z$$

$$\psi_m = B\,(\pi a^2) = \mu_o \mu_r \frac{N\,I}{l}\,(\pi a^2)$$

$$L = \frac{\Lambda}{I} = \frac{N\,\psi_m}{I} = \mu_o \mu_r \frac{N^2}{l}\,(\pi a^2) \qquad [H]$$

Example:

Find the self inductance of the torrid shown.



Solution:

$$L = \frac{\Lambda}{I} = \frac{N\,\psi_m}{I}$$

The magnetic field intensity $\vec{H}$ can be determined by applying Ampere's circuital law as follows:

(1) $\oint \vec{H}\,.\,\vec{dl} = I_{en}$

(2) *Choice of Amperian loop*

(3) $I_{en} = N\,I$

(4) $\oint \vec{H}\,.\,\vec{dl} = H\,2\pi r_c$

(5) $\vec{H} = \frac{N\,I}{2\pi r_c}\,\hat{a}_\varphi$

So,

$$\vec{B} = \mu_o\mu_r\vec{H} = \mu_o\mu_r\frac{N\,I}{2\pi r_c}\,\hat{a}_\varphi$$

Average path radius $r_{c|average} = b = \frac{a+c}{2}$

$$\vec{B}_{av} = \psi_m = \hat{a}_\varphi$$

$$\psi_m = B_{av}\,A$$

$$A = \pi\left(\frac{c-a}{2}\right)^2$$

$$\psi_m = \mu_o\mu_r\frac{N\,I}{2\pi b}\,\pi\left(\frac{c-a}{2}\right)^2$$

$$\psi_m = \mu_o\mu_r\frac{N\,I(c-a)^2}{4(c+a)}$$

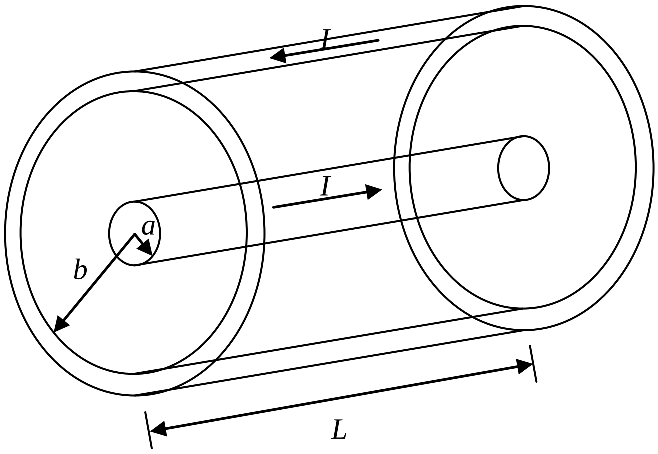
$$L = \frac{\Lambda}{I} = \frac{N\,\psi_m}{I} = \mu_o\mu_r\frac{N^2(c-a)^2}{4(c+a)} \quad \textit{Henery}$$

Example:

Find the self inductance per unit length of a T.L. (coaxial cable).

Solution:

$$L = \frac{\Lambda}{I} = \frac{N\,\psi_m}{I} \qquad N = 1$$



The magnetic field intensity $\vec{H}$ can be determined by applying Ampere's circuital law as follows:

(1) $\oint \vec{H} \cdot \vec{dl} = I_{en}$

(2) $Choice\ of\ Amperian\ loop$

(3) $I_{en} = I$

(4) $\oint \vec{H} \cdot \vec{dl} = H\, 2\pi r_c$

(5) $\vec{H} = \frac{I}{2\pi r_c}\, \hat{a}_\varphi$

So,

$$\vec{B} = \mu_o \mu_r \vec{H} = \mu_o \mu_r \frac{I}{2\pi r_c}\, \hat{a}_\varphi$$

$$\psi_m = \iint \vec{B} \cdot \vec{ds} = \int_0^l \int_a^b \mu_o \mu_r \frac{I}{2\pi r_c}\, \hat{a}_\varphi . \hat{a}_\varphi\, dr_c dz$$

$$\psi_m = \frac{\mu_o \mu_r I l}{2\pi} \ln(r_c)_a^b = \frac{\mu_o \mu_r I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

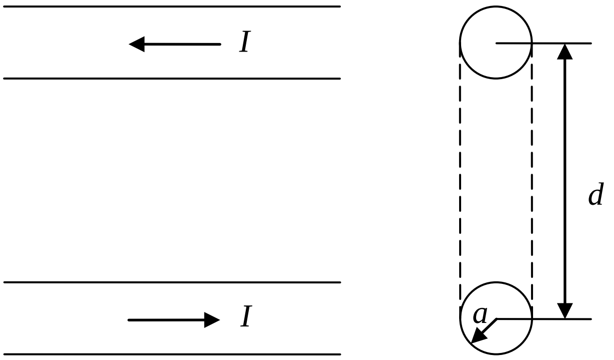
$$L = \frac{\Lambda}{I} = \frac{\psi_m}{I} = \frac{\mu_o \mu_r l}{2\pi} \ln\left(\frac{b}{a}\right) \qquad [H]$$

$$\frac{L}{l} = \frac{\mu_o \mu_r}{2\pi} \ln\left(\frac{b}{a}\right) \qquad [H/m]$$

Example:

Find the inductance per meter for two parallel wires.

Solution:



$$L = \frac{\Lambda}{I} = \frac{N\,\psi_m}{I} \qquad N = 1$$

The magnetic field intensity $\vec{H}$ can be determined by applying Ampere's circuital law as follows:

(1) $\oint \vec{H}\,.\,\vec{dl} = I_{en}$

(2) $Choice\ of\ Amperian\ loop$

(3) $I_{en} = I$

(4) $\oint \vec{H}\,.\,\vec{dl} = H\,2\pi r_c$

(5) $\vec{H} = \frac{I}{2\pi r_c}\,\hat{a}_\varphi$

So,

$$\vec{B} = \mu_o \mu_r \vec{H} = \mu_o \mu_r \frac{I}{2\pi r_c}\,\hat{a}_\varphi$$

$$\psi_m = \iint \vec{B}\,.\,\vec{ds} = \int_0^l \int_a^{d-2a} \mu_o \mu_r \frac{I}{2\pi r_c}\,\hat{a}_\varphi . \hat{a}_\varphi dr_c dz$$

$$\psi_m = \frac{\mu_o \mu_r I l}{2\pi} \ln(r_c)_a^{d-2a} = \frac{\mu_o \mu_r I l}{2\pi} \ln\left(\frac{d-2a}{a}\right)$$

$$L = \frac{\Lambda}{I} = \frac{\psi_m}{I} = \frac{\mu_o \mu_r l}{2\pi} \ln\left(\frac{d-2a}{a}\right) \qquad [H]$$

Mutual inductance:

Mutual inductance exists between two magnetic circuits that share a common flux linkage. So, the mutual inductance is due to the influence of one circuit on another and vice versa.

$$M_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2\, \psi_{m12}}{I_1}$$

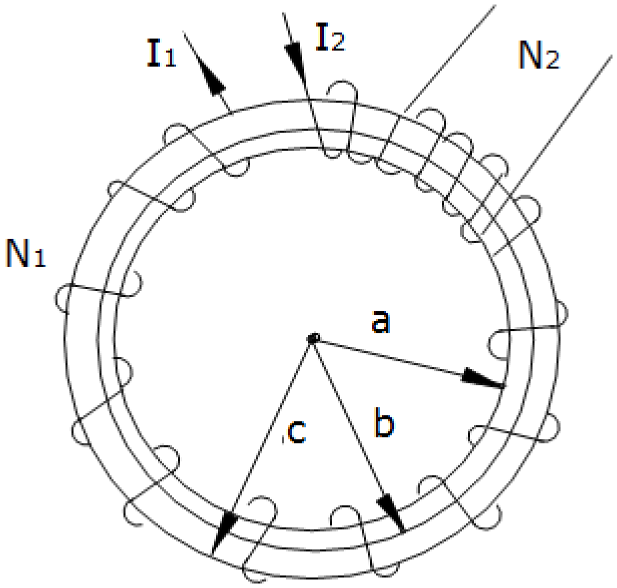
Where, Λ_{12} is the linkage of circuit 2 produced by I_1 in circuit 1.

For a linear magnetic medium, it can be shown that $M_{12} = M_{21}$.

$$M_{21} = \frac{\Lambda_{21}}{I_2} = \frac{N_1\, \psi_{m21}}{I_2}$$

Example:

Find M_{12}.



Solution:

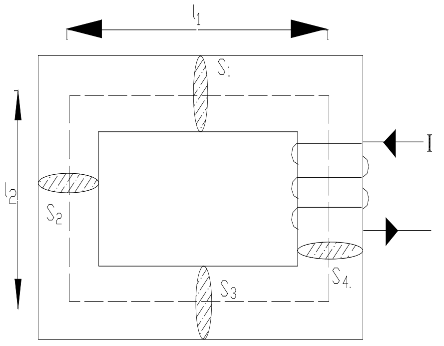
$$\psi_{12} = \frac{\mu_0 \mu_r N_1 I_1 (c-a)^2}{4(c+a)}$$

$$M_{12} = \frac{N_2 \psi_{12}}{I_1}$$

$$\therefore M_{12} = \frac{\mu_0 \mu_r N_1 N_2 I_1 (c-a)^2}{4 I_1 (c+a)} = \frac{\mu_0 \mu_r N_1 N_2 (c-a)^2}{4(c+a)}$$

Magnetic Circuits:

Example:



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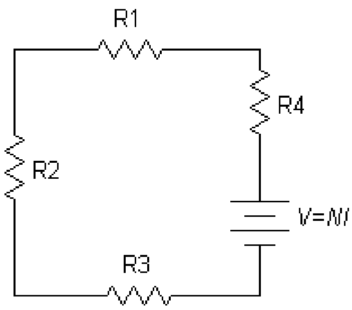


$$NI = R\psi_m$$

$$\text{R} = \text{Reluctance}$$

$$= \text{The impedance of magnetic material for flow of magnetic flux}$$

$$R = \frac{l}{\mu_0 \mu_r S} \quad \rightarrow \quad l = \text{magnetic path length , S = cross sectional area}$$

$$R_1 = \frac{l_1}{\mu_0 \mu_r S_1},$$

$$R_2 = \frac{l_2}{\mu_0 \mu_r S_2},$$

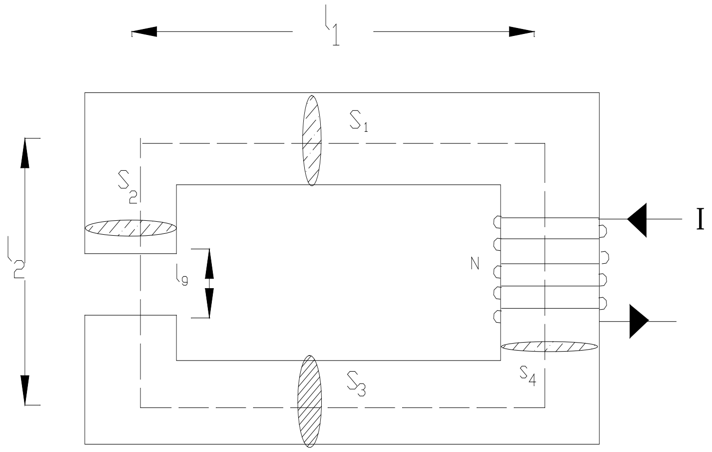
$$R_3 = \frac{l_3}{\mu_0 \mu_r S_3},$$

$$R_4 = \frac{l_{4_1}}{\mu_0 \mu_r S_4}$$

$$\therefore \psi_m \left(R_1 + R_2 + R_3 + R_4\right) = NI$$

$$\therefore \psi_m = \frac{NI}{R_1 + R_2 + R_3 + R_4} = B \cdot A = B \cdot S$$

Example:

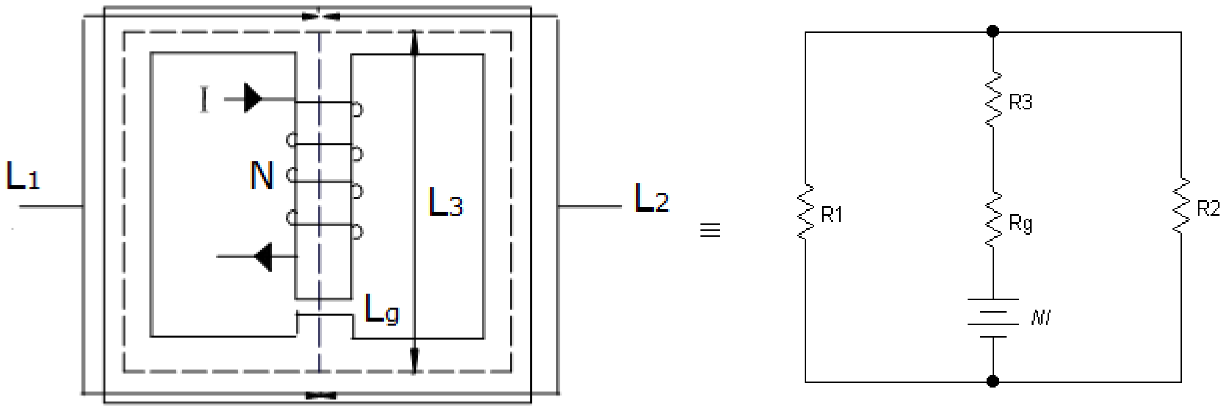


$$R_g = \frac{l_g}{\mu_0 S_2}$$

$$\therefore \psi_m \left(R_1 + R_2 + R_3 + R_4 + R_g\right) = NI$$

$$\psi_m = \frac{NI}{\left(R_1 + R_2 + R_3 + R_4 + R_g\right)} = B.S$$

Example:



$$\psi_{m3} = \psi_{m1} + \psi_{m2}$$

$$R_t = R_g + R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

$$\psi_{m3} \cdot R_t = NI \quad \rightarrow \quad \psi_{m3} = \frac{NI}{R_t}$$

$$\therefore \psi_{m1} = \psi_{m3} \cdot \frac{R_2}{R_1 + R_2} \quad ,$$

$$\psi_{m2} = \psi_{m3} \cdot \frac{R_1}{R_1 + R_2}$$