Australian Journal of Basic and Applied Sciences, 4(8): 3803-3814, 2010
ISSN 1991-8178

A Generalized Thermoelasticity Problem for a Half Space with Heat Sources under Axisymmetric Distributions

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Abstract: In this work, we solve a transient two dimensional problem for an infinite thermoelastic half space whose surface is traction free and subjected to a known axisymmetric temperature distribution. The problem is considered within the context of the theory of generalized thermoelasticity with one relaxation time. Axisymmetric heat sources permeate the medium. The problem is solved using the Laplace and Hankel transforms. The solution in the transformed domain is obtained by using a direct approach. The inverse transforms are obtained using a numerical technique. Numerical results for the temperature, stress and displacement distributions are obtained and represented graphically.

Key words:

INTRODUCTION

The theory of dynamic thermoelasticity is of much importance in various engineering fields such as earthquake engineering, nuclear reactors' design, high-energy particle accelerators, etc. The theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body was introduced by Lord and Shulman (1967). This theory was extended by Dhaliwal and Sherief (1980) to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and
hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. Uniqueness of solution for this theory was proved under different conditions by Ignaczak (1979), by Dhaliwal and Sherief (1980) and by Sherief (1987). The state space approach to this theory was developed by Anwar and Sherief (1988) and by Sherief (1993) for one-dimensional problems and by Sherief and Anwar (1994) for two-dimensional problems. The fundamental solution for this theory was obtained by Sherief (1986) and by Sherief and Anwar (1986). Sherief and Hamza (1994) have solved a two-dimensional problem for a thick layer and studied wave propagation in this theory; they have obtained the solution for two dimensional axisymmetric problems in spherical regions in (Sherief, H. and F. Hamza, 1996). Sherief and El-Maghraby have solved two problems including cracks in (Sherief, H. and N. El-Maghraby, 2003; Sherief, H. and N. El-Maghraby, 2005). Sherief and Helmy have solved a two-dimensional problem in (Sherief, H. and K. Helmy, 1999). A two-dimensional problem for a half-space and for a thick plate with heat sources have been solved by El-Maghraby in (2004, 2005). A problem for a half-space under the action of a body force has been solved by Heba Saleh in (2005). A two-dimensional problem for a half-space and for a thick plate under the action of body forces have been solved by El-Maghraby in (2008, 2009).

Formulation of the Problem:

We shall consider a homogeneous isotropic thermoelastic solid occupying the region $z \geq 0$. The z-axis is taken perpendicular to the bounding plane pointing inwards. The problem is considered within the context of the theory of generalized thermoelasticity with one relaxation time. We shall assume that the initial state of the medium is quiescent. The surface of the medium is traction free and subjected to a known axisymmetric temperature distribution. Axisymmetric heat sources permeate the medium. Cylindrical polar coordinates (r, j, z) are used.

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The problem is thus two-dimensional with all functions considered depending on the spatial variables r and z as well as on the time variable t.

The displacement vector, thus, has the form

$$\underline{\mathbf{u}} = (u, 0, w)$$

The equations of motion can be written as (Sherief, H. and N. El-Maghraby, 2003)

$$\mu \nabla^2 u - \frac{\mu}{r^2} u + (\lambda + \mu) \frac{\partial e}{\partial r} - \gamma \frac{\partial T}{\partial r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\mu \nabla^2 w + (\lambda + \mu) \frac{\partial e}{\partial z} - \gamma \frac{\partial T}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{2}$$

The generalized equation of heat conduction has the form (Sherief, H. and N. El-Maghraby, 2003)

$$k \nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) (\rho c_E T + \gamma T_0 e) - \rho \left(1 + \tau_0 \frac{\partial}{\partial t} \right) Q \tag{3}$$

In the above equations, T is the absolute temperature and e is the cubical dilatation given by the relation (Sherief, H. and N. El-Maghraby, 2003)

$$e = \frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} (r u) + \frac{\partial w}{\partial z} \tag{4}$$

while r is the density, l and m are Lamé's constants, k is the thermal conductivity, g is a material constant given by g = (3l+2m) a_t, a_t being the coefficient of linear thermal expansion, T_0 is a reference temperature assumed to be such that $|(T-T_0)/T_0| << 1$ and c_E is the specific heat at constant strain, t_0 is a constant with the dimensions of time that acts as a relaxation time. $\tilde{N}^2$ is Laplace's operator, given in our case by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The following constitutive relations supplement the above equations

$$\sigma_{rr} = 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma \left(T - T_0 \right) \tag{5.a}$$

$$\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda e - \gamma \left(T - T_0 \right) \tag{5.b}$$

$$\sigma_{rz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \tag{5.c}$$

We shall use the following non-dimensional variables

$$r' = c_1 \eta r \ , \ z' = c_1 \eta z \ , \ u' = c_1 \eta u \ , \ w' = c_1 \eta w \ , \ t' = c_1^2 \eta t$$

$$\tau_0^{'} = c_1^2 \eta \tau_0 \ , \ \sigma_{ij}^{'} = \frac{\sigma_{ij}}{\mu} \ , \ \theta = \frac{\gamma\left(T - T_0\right)}{(\lambda + 2\mu)}, \ \ Q^{'} = \frac{\rho \gamma Q}{k\, c_1^2 \eta^2 \, (\lambda + 2\mu)}$$

where $\eta = \dfrac{\rho c_E}{k}$, $c_1 = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$. c_1 is the speed of propagation of isothermal elastic waves.

Using the above non-dimensional variables, the governing equations take the form (dropping the primes for convenience)

$$\nabla^2 u - \frac{u}{r^2} + (\beta^2 - 1)\frac{\partial e}{\partial r} - \beta^2 \frac{\partial \theta}{\partial r} = \beta^2 \frac{\partial^2 u}{\partial t^2} \tag{6}$$

$$\nabla^2 w + (\beta^2 - 1)\frac{\partial e}{\partial z} - \beta^2 \frac{\partial \theta}{\partial z} = \beta^2 \frac{\partial^2 w}{\partial t^2} \tag{7}$$

$$\nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right)(\theta + \varepsilon\, e) - \left(1 + \tau_0 \frac{\partial}{\partial t} \right) Q \tag{8}$$

while the constitutive relations (5.a)-(5.c) become

$$\sigma_{rr} = 2\frac{\partial u}{\partial r} + (\beta^2 - 2)\, e - \beta^2\, \theta \tag{9.a}$$

$$\sigma_{zz} = 2\frac{\partial w}{\partial z} + (\beta^2 - 2)\, e - \beta^2\, \theta \tag{9.b}$$

$$\sigma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{9.c}$$

We note that equation (4) retains its form.

In the above equations $\beta^2 = \dfrac{\lambda + 2\,\mu}{\mu}$.

Combining equations (6)-(8), we obtain upon using equation (4)

$$\nabla^2 e - \nabla^2 \theta = \frac{\partial^2 e}{\partial t^2} \tag{10}$$

The boundary conditions for the heat conduction problem at z = 0 may be taken as:

q(r,0,t) = f(r,t) , 0 < r < ¥ , (11.a)

s_{zz} (r,0,t) = 0, 0 < r < ¥ , (11.b)

s_{rz} (r,0,t) = 0 , 0 < r < ¥ , (11.c)

where f(r,t) is a known function of r and t.

Solution in the Transformed Domain:

Applying the Laplace transform defined by the relation

$$\overline{f}(r,z,s)=\mathrm{L}[f(r,z,t)]=\int_0^\infty e^{-st}\, f(r,z,t)\,dt$$

to both sides of equations (6)-(10), we obtain

$$\nabla^2\overline{u}-\frac{\overline{u}}{r^2}+(\beta^2-1)\frac{\partial\overline{e}}{\partial r}-\beta^2\frac{\partial\overline{\theta}}{\partial r}=\beta^2 s^2\,\overline{u} \tag{12}$$

$$\nabla^2\overline{w}+(\beta^2-1)\frac{\partial\overline{e}}{\partial z}-\beta^2\,\frac{\partial\overline{\theta}}{\partial z}=\beta^2\,s^2\,\overline{w} \tag{13}$$

$$(\nabla^2-s-\tau_0 s^2)\,\overline{\theta}=\left(1+\tau_0 s\right)(\varepsilon s\,\overline{e}-\overline{Q}) \tag{14}$$

$$(\nabla^2-s^2)\,\overline{e}=\nabla^2\overline{\theta} \tag{15}$$

$$\overline{\sigma}_{rr}=2\frac{\partial\overline{u}}{\partial r}+(\beta^2-2)\,\overline{e}-\beta^2\,\overline{\theta} \tag{16.a}$$

$$\overline{\sigma}_{zz}=2\frac{\partial\overline{w}}{\partial z}+(\beta^2-2)\,\overline{e}-\beta^2\,\overline{\theta} \tag{16.b}$$

$$\overline{\sigma}_{rz}=\frac{\partial\overline{u}}{\partial z}+\frac{\partial\overline{w}}{\partial r} \tag{16.c}$$

The boundary conditions (11), in the transformed domain, take the form

$$\overline{\theta}(r,0,s)=\overline{f}(r,s)\ ,\quad 0<r<\infty\,, \tag{17.a}$$

$$\overline{\sigma}_{zz}\,(r,0,s)=0\,,\ \ 0<r<\infty \tag{17.b}$$

$$\overline{\sigma}_{rz}\,(r,0,s)=0\,,\ \ 0<r<\infty \tag{17.c}$$

Eliminating $\overline{e}$ between equations (14) and (15), we get

$$\left[\nabla^4-\left(s^2+s\,(1+\tau_0 s)(1+\varepsilon)\right)\nabla^2+s^3\left(1+\tau_0 s\right)\right]\overline{\theta}=-\left(1+\tau_0 s\right)\left(\nabla^2-s^2\right)\overline{Q} \tag{18}$$

The above equation can be factorized as

$$\left(\nabla^2-k_1^2\right)\left(\nabla^2-k_2^2\right)\overline{\theta}\;=\;-\left(1+\tau_0 s\right)\left(\nabla^2-s^2\right)\overline{Q}\ , \tag{19}$$

where k_1^2 and k_2^2 are the roots with positive real parts of the characteristic equation

$$k^4 - \left(s^2 + s(1+\tau_0 s)(1+\varepsilon)\right) k^2 + s^3\left(1+\tau_0 s\right) = 0 \tag{20}$$

The solution of equation (19) can be written in the form $\overline{\theta} = \overline{\theta}_1 + \overline{\theta}_2 + \overline{\theta}_p$ where q_i is the solution of the homogenous equation

$$(\nabla^2 - k_i^2)\overline{\theta}_i = 0 \quad , i = 1,2. \tag{21}$$

and $\overline{\theta}_p$ is a particular solution of equation (19).

In order to solve the above equation, we shall use the Hankel transform of order zero with respect to r.

This transform of a function $\overline{f}(r,z,s)$ is defined by the relation (Sneddon, I.N., 1995; Churchill, R.V., 1972)

$$\overline{f}^*(\alpha,z,s) = \mathrm{H}[\overline{f}(r,z,s)] = \int_0^\infty \overline{f}(r,z,s)\, r\, J_0(\alpha r)\, dr$$

where J_0 is the Bessel function of the first kind of order zero.

The inverse Hankel transform is given by the relation (Sneddon, I.N., 1995; Churchill, R.V., 1972)

$$\overline{f}(r,z,s) = \mathrm{H}^{-1}[\overline{f}^*(\alpha,z,s)] = \int_0^\infty \overline{f}^*(\alpha,z,s)\, \alpha\, J_0(\alpha r)\, d\alpha$$

Applying the Hankel transform with parameter a to both sides of equation (21) and using the following operational relation of the Hankel transform (Churchill, R.V., 1972)

$$\mathrm{H}\left(\frac{\partial^2 \overline{f}(r,z,s)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{f}(r,z,s)}{\partial r}\right) = -\alpha^2 \overline{f}^*(\alpha,z,s) .$$

we obtain

$$\left[\mathrm{D}^2 - (k_i^2 + \alpha^2)\right]\overline{\theta}_i^* = 0, \;\; i = 1,2 \;\; , \text{ where } \;\mathrm{D} = \partial/\partial z$$

The solution of the above equation, which is bounded at infinity, can be written as

$$\overline{\theta}_i^*(\alpha,z,s) = A_i(\alpha,s)\left(k_i^2 - s^2\right)e^{-q_i z} \quad , \text{ where } \; q_i = \sqrt{\alpha^2 + k_i^2}$$

Applying the Hankel transform to both sides of equation (19) and using equation (21), we obtain

$$(\mathrm{D}^2 - q_1^2)(\mathrm{D}^2 - q_2^2)\overline{\theta}_p^* = -\left(1+\tau_0 s\right)\left(\mathrm{D}^2 - q^2\right)\overline{Q}^*, \tag{22}$$

where $q = \sqrt{\alpha^2 + s^2}$.

From now on, we shall take the heat source in the form

$$Q(r,z,t) = \frac{H(t)\sqrt{r}\, e^{-z}}{\sqrt{1+r^2}}$$

where H(.) is the Heaviside unit step function.

Thus, $\bar{Q}^*(\alpha,z,s) = \dfrac{e^{-(\alpha+z)}}{s\sqrt{\alpha}}$.

The solution of equation (22) has the form

$$\bar{\theta}_p^* = -\frac{(1+\tau_0 s)(1-q^2)\,e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)}$$

Then the complete solution of equation (19) in the transformed domain can be written as

$$\bar{\theta}^*(\alpha,z,s) = \sum_{i=1}^{2} A_i(\alpha,s)\left(k_i^2 - s^2\right)e^{-q_i z} - \frac{(1+\tau_0 s)(1-q^2)\,e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)}$$

Taking the inverse Hankel transform of both sides of the above equation, we obtain

$$\bar{\theta}(r,z,s) = \int_0^\infty \left\{ \sum_{i=1}^{2} A_i(\alpha,s)\left(k_i^2 - s^2\right)e^{-q_i z} - \frac{(1+\tau_0 s)(1-q^2)\,e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)} \right\} \alpha J_0(\alpha r)\,d\alpha. \tag{23}$$

Similarly eliminating $\bar{\theta}$ between equations (14) and (15), we find that $\bar{e}$ satisfies the differential equation

$$\left(\nabla^2 - k_1^2\right)\left(\nabla^2 - k_2^2\right)\bar{e} = -\left(1+\tau_0 s\right)\nabla^2 \bar{Q} \quad .$$

The solution of this equation compatible with equations (15) and (23) is given by

$$\bar{e}^*(\alpha,z,s) = \sum_{i=1}^{2} A_i(\alpha,s)\,k_i^2\, e^{-q_i z} - \frac{(1+\tau_0 s)(1-\alpha^2)\,e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)}$$

Using the inverse Hankel transform, we obtain

$$\bar{e}(r,z,s) = \int_0^\infty \left\{ \sum_{i=1}^{2} A_i(\alpha,s)\,k_i^2\, e^{-q_i z} - \frac{(1+\tau_0 s)(1-\alpha^2)\,e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)} \right\} \alpha\, J_0(\alpha r)\,d\alpha. \tag{24}$$

Applying the Hankel transform to both sides of equation (13), we obtain upon using equations (23) and (24)

$$\left(D^2-\alpha^2-\beta^2 s^2\right)\overline{w}^* = -\sum_{i=1}^{2}(k_i^2-\beta^2 s^2)A_i\, q_i\, e^{-q_i z} - \frac{(1+\tau_0 s)(\beta^2 s^2+\alpha^2-1)\, e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)}. \tag{25}$$

The solution of equation (25), which is bounded at infinity is given by

$$\overline{w}^*(\alpha,z,s) = B(\alpha,s)\, e^{-q_3 z} - \sum_{i=1}^{2} A_i(\alpha,s) q_i e^{-q_i z} + \frac{(1+\tau_0 s)\; e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)} \;,$$

where B(a,s) is a parameter and $q_3 = \sqrt{\alpha^2+\beta^2 s^2}$.

Taking the inverse Hankel transform of both sides of the above equation, we obtain

$$\overline{w}(r,z,s) = \int_0^\infty \left\{ B(\alpha,s)e^{-q_3 z} - \sum_{i=1}^{2} A_i(\alpha,s) q_i e^{-q_i z} + \frac{(1+\tau_0 s)\; e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)} \right\} \alpha J_0(\alpha r)\, d\alpha. \tag{26}$$

Taking both the Laplace and the Hankel transforms of both sides of equation (4), we obtain upon using equations (24) and (26)

$$H\left[\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u})\right] = B(\alpha,s)\, q_3\; e^{-q_3 z} - \alpha^2\left(\sum_{i=1}^{2} A_i(\alpha,s)e^{-q_i z} - \frac{(1+\tau_0 s)\; e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)}\right).$$

It is not difficult to show that the solution of this equation is given by

$$\overline{u}(r,z,s) = \int_0^\infty \left\{ B(\alpha,s) q_3\; e^{-q_3 z} - \alpha^2\left[\sum_{i=1}^{2} A_i(\alpha,s) e^{-q_i z} - \frac{(1+\tau_0 s)\; e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)}\right]\right\} J_1(\alpha r)\, d\alpha. \tag{27}$$

Substituting from equations (23), (24), (26) and (27) into equations (16.b) and (16.c), we obtain upon using the equation (Churchill, R.V., 1972)

$$\frac{d\,J_0(z)}{d\,z} = -J_1(z) \;,$$

the stress tensor components in the form

$$\bar{\sigma}_{zz} = \int_0^\infty \left\{ -2q_3 B(\alpha,s) e^{-q_3 z} + \left(\alpha^2 + q_3^2\right) \left[\sum_{i=1}^{2} A_i(\alpha,s) e^{-q_i z} - \frac{(1+\tau_0 s)\, e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)} \right] \right\} \alpha J_0(\alpha r)\, d\alpha, \tag{28}$$

$$\bar{\sigma}_{rz} = \int_0^\infty \left\{ -\left(\alpha^2 + q^2\right) B(\alpha,s) e^{-q_3 z} + 2\alpha^2 \left[\sum_{i=1}^{2} A_i q_i(\alpha,s) e^{-q_i z} - \frac{(1+\tau_0 s)\, e^{-(\alpha+z)}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)} \right] \right\} J_1(\alpha r)\, d\alpha. \tag{29}$$

Appling the Hankel transform to the boundary conditions (17), we obtain

$$\bar{\theta}^*(\alpha,0,s) = \bar{f}^*(\alpha,s) \tag{30}$$

$$\bar{\sigma}_{zz}^*(\alpha,0,s) = 0, \tag{31}$$

$$\bar{\sigma}_{rz}^*(\alpha,0,s) = 0. \tag{32}$$

We shall now apply the boundary conditions (30)-(32) to determine the unknown parameters, we have

$$\sum_{i=1}^{2} A_i(\alpha,s)\left(k_i^2 - s^2\right) = \frac{(1+\tau_0 s)(1-q^2)\, e^{-\alpha}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)} + \bar{f}^*(\alpha,s), \tag{33}$$

$$\left(\alpha^2 + q_3^2\right) \sum_{i=1}^{2} A_i(\alpha,s) - 2q_3 B(\alpha,s) = \frac{(1+\tau_0 s)\left(\alpha^2 + q_3^2\right) e^{-\alpha}}{s\sqrt{\alpha}\,(1-q_1^2)(1-q_2^2)}, \tag{34}$$

$$2\alpha^2 \sum_{i=1}^{2} A_i(\alpha,s) q_i - \left(\alpha^2 + q^2\right) B(\alpha,s) = \frac{2\alpha\sqrt{\alpha}\,(1+\tau_0 s)\, e^{-\alpha}}{s\,(1-q_1^2)(1-q_2^2)}. \tag{35}$$

Solving equations (33)-(35) numerically, we get the complete solution of the problem in the transformed domain.

Inversion of the Double Transform:

We shall now outline the numerical inversion method used to find the solution in the physical domain.

Let $\bar{f}^*(\alpha,z,s)$ be the Hankel-Laplace transform of a function *f(r,z,t)* . The complex inversion formula for Laplace transforms can be written as (Churchill, R.V., 1972)

$$f^*(\alpha,z,t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\,\bar{f}^*(\alpha,z,s)ds \ ,$$

where *d* is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}^*(\alpha,z,s)$. Taking s = d + i y, the above integral takes the form

$$f^*(\alpha,z,t) = \frac{e^{dt}}{2\pi}\int_{-\infty}^{\infty} e^{ity}\,\bar{f}^*(\alpha,z,d+iy)dy$$

Expanding the function *h(a,z,t)* = exp*(-dt)f(a,z,t)* in a Fourier series in the interval [0,2T], we obtain the approximate formula (Honig, G. and U. Hirdes, 1984)

$$f^*(\alpha,z,t) = f_\infty(\alpha,z,t) + E_D \ ,$$

where

$$f_\infty(\alpha,z,t) = \tfrac{1}{2}c_0(\alpha,z,t) + \sum_{k=1}^{\infty} c_k(\alpha,z,t) \qquad \text{for } 0 \le \mathrm{t} \le 2\mathrm{T} \tag{36}$$

and

$$c_k(\alpha,z,t) = \frac{e^{dt}}{T}\mathrm{Re}\left[e^{ik\pi t/T}\,\bar{f}^*(\alpha,z,d+ik\pi/T)\right] \ , \ k = 0,1,2,\ldots \tag{37}$$

E_D , the discretization error, can be made arbitrarily small by choosing the constant *d* large enough (Honig, G. and U. Hirdes, 1984).

Since the infinite series in equation (36) can only be summed up to a finite number *N* of terms, the approximate value of *f(a,z,t)* becomes

$$f_N(\alpha,z,t) = \tfrac{1}{2}c_0(\alpha,z,t) + \sum_{k=1}^{N} c_k(\alpha,z,t) \qquad \text{for } 0 \le t \le 2T \tag{38}$$

Using the above formula to evaluate *f(a,z,t)*, we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the `Korrecktur` method (Honig, G. and U. Hirdes, 1984) is used to reduce the discretization error. Next, the e-algorithm is used to reduce the truncation error and hence to accelerate convergence.

The Korrecktur method uses the following formula to evaluate the function $f^*(\alpha,z,t)$

$$f^*(\alpha,z,t) = f_\infty(\alpha,z,t) - e^{-2dT} f_\infty(\alpha,z,2T+t) + E'_D \ ,$$

where the discretization error $|E'_D| << |E_D|$ (Honig, G. and U. Hirdes, 1984). Thus, the approximate value of $f^*(\alpha,z,t)$ becomes

$$f_{NK}(\alpha,z,t) = f_N(\alpha,z,t) - e^{-2dT} f_{N'}(\alpha,z,2T+t) \tag{39}$$

$N`$ is an integer such that $N' < N$.

We shall now describe the e-algorithm that is used to accelerate the convergence of the series in equation (38). Let N be an odd natural number, and let

$$s_m(r,z,t) = \sum_{k=1}^{m} c_k(r,z,t)$$

be the sequence of partial sums of (38). We define the e-sequence by

$$\varepsilon_{0,m} = 0, \varepsilon_{1,m} = s_m$$

and

$e_{p+1,m} = e_{p-1,m+1} + 1/(e_{p,m+1} - e_{p,m})$, p =1,2,3,...

It can be shown that (Honig, G. and U. Hirdes, 1984) the sequence

$e_{1,1}$, $e_{3,1}$, $e_{5,1}$, ... , $e_{N,1}$,

converges to $f^*(\alpha,z,t)$ + E_D - $c_0/2$ faster than the sequence of partial sums

s_m , m = 1,2,3,...

The actual procedure used to invert the Laplace transforms consists of using equation (39) together with the e-algorithm. The values of d and T are chosen according to the criteria outlined in (Honig, G. and U. Hirdes, 1984).

Next, we invert the Hankel transform $f^*(\alpha,z,t)$ using numerical integration techniques to obtain the required function $f(r,z,t)$.

Numerical Results:

In what follows we shall take

$$f(r,t) = \theta_0 H(a-r) H(t),$$

where θ_0 is a constant. This means that the surface of the half space is suddenly heated to the temperature θ_0 at the start inside a circle of radius "a" and a center at the origin. The rest of the surface is kept at a zero temperature.

We thus have (Churchill, R.V., 1972)

$$\overline{f}^*(\alpha,s) = \frac{-a\theta_0 J_1(\alpha a)}{\alpha s}$$

The copper material was chosen for purposes of numerical evaluations. The constants of the problem are shown in table 1

Table 1:

$k = 386$	$a_t = 1.78\ (10)^{-5}$	$c_E = 383.1$	$h = 8886.73$
$m = 3.86\ (10)^{10}$	$l = 7.76\ (10)^{10}$	$r = 8954$	$c_1 = 4.158(10)^3$
$b^2 = 4$	$T_0 = 293$	$c = 0.01$	$g = 1.61$
$e = 0.0168$	$t_0 = 0.02$	$a = 1$	$q_0 = 1$

The computations were performed for three values of non-dimensional time, namely for t = 0.05, t = 0.1 and t = 0.15. The numerical technique outlined above was used to obtain the temperature, the axial displacement and the axial stress distributions. The temperature increment q is represented by the graph in figure 1. The displacement component w is shown in figure 2, the axial stress component s_{zz} is shown in figure 3.

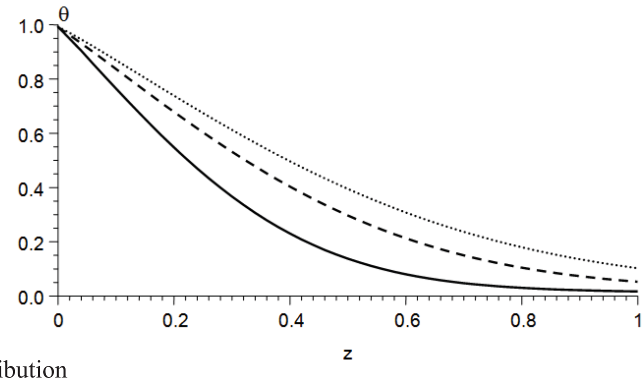

Fig. 1: Temperature Distribution

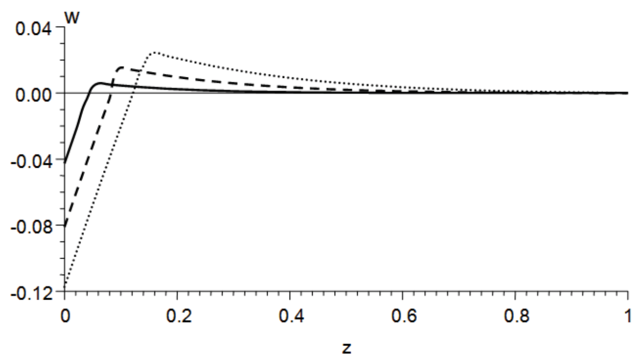

Fig. 2: Displacement Distribution

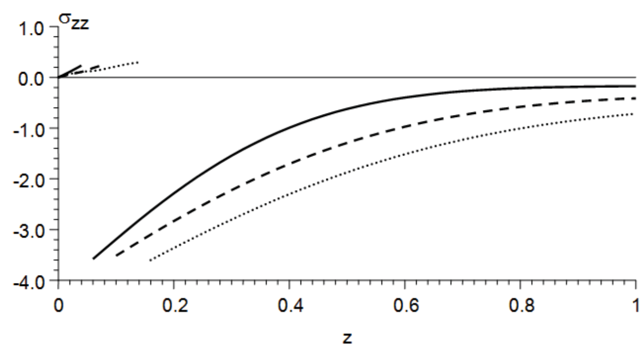

Fig. 3: Stress Distribution

All the definite integrals involved were calculated using Romberg technique of numerical integration with variable step size.

It was found that the inclusion of the heat sources has a significant effect on both the temperature and the stress. Its effect on the displacement is very small

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