Applied Mathematical Sciences, Vol. 1, 2007, no. 24, 1169 - 1180

Lie-Group Method for Predicting Water Content for Immiscible Flow of Two Fluids in a Porous Medium

Mina B. Abd-el-Malek [a,*,1] , Nagwa A. Badran [a], Hossam S. Hassan [b]

[a] Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt

[b] Department of Basic and Applied Sciences, Arab Academy for Science and Technology and Maritime Transport, P.O.BOX 1029 Alexandria, Egypt

Abstract. We apply Lie-group method for determining symmetry reductions to nonlinear diffusion-convection equation arising in modeling the immiscible flow of two fluids in a porous medium. This equation was presented by Fokas and Yortsos [1], for oil and water flow in petroleum-reservoir, which is given by

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right] - V(t)\frac{\partial f(\theta)}{\partial x},$$

where t is time, x is the space variable, θ is the normalized volumetric water content ($0 \le \theta \le 1$), $D(\theta)$ is the water diffusivity, $f(\theta)$ is the fractional flow function and $V(t)$ is the combined flow rate of both fluids which is a function of time only, since we assume incompressibility of both fluids. Lie-group method starts out with a general infinitesimal group of transformations under which a given partial differential equation is invariant, then, the determining equations are derived. The determining equations are a set of linear differential equations, the solution of which gives the transformation functions or the infinitesimals of the dependent and independent variables. After the group has been established, a solution to the given partial differential equation may be found from the invariant surface condition such that its solution leads to similarity variables which reduce the number of independent variables in the given partial differential equation.

Mathematics Subject Classification: 22E05, 35Q53, 54H15
Keywords: Diffusion-convection equation, Lie-group method, Porous medium

---

* Corresponding author, Tel.: +20-3-391-8715; Fax: +20-2-795-7565.
E-mail addresses: minab@aucegypt.edu (Mina B.Abd-el-Malek), hossams@aast.edu (Hossam S.Hassan).

[1] Present address: Department of Mathematics, The American University in Cairo, Cairo 11511, Egypt

1 Introduction

Multiphase flow processes in natural porous media or industrial synthetic porous materials arise in many diverse fields of science and engineering, such as agricultural, biomedical, ceramic, chemical, and petroleum engineering, food and soil sciences and powder metallurgy.

Typical examples of great practical interest and economic importance are the production of oil and natural gas from reservoir rocks, the groundwater flow and the soil remediation through the removal of liquid organic pollutants.

Petroleum reservoir simulators help oil companies to make effective use of expensive data colleted through field's measurements, data processing and interpretation.

Two-phase flow modeling is concerning the displacement of one fluid, say oil, by another, say water, within a reservoir. This model problem may be characterized by the injection of a wetting fluid (water) into the reservoir at a particular location, displacing the non-wetting fluid (oil), which is being withdrawn at another location.

Due to the physical interaction between the two phases, water and oil, this production process leads to a moving shock front at the interface between the two phases. The evaluation of the shock front is of primary importance for the production, where the goal is to withdraw as much oil as possible before the breakthrough, when water arrives at the production site. However, the geometry of the moving shock front may be very complicated, mainly due to varying flow velocities.

Water flooding is one effective technique for the exploration and production of hydrocarbons from petroleum reservoirs. This technique involves drilling wells into the rocks, injectors and producers. By the injection of water through the injectors, hydrocarbons are then, due to the resulting pressure, pushed into the rocks and forced to flow towards the producers.

In this work, we consider as an example the petroleum industry, where one of the fluids would be water and the other would be one of a variety of hydrocarbon liquids (oil). The difficulty of this problem is the highly non-linear nature of the term arising from the so-called capillary drive [13].

In 1982, Fokas and Yortsos [1] obtained a closed form, analytic solution for one-dimensional flow with capillary drive under the assumption that the capillary hydraulic functions have the forms $D(\theta)=\frac{D_0}{(1-\nu\theta)^2}$ and $f(\theta)=\frac{(1-\nu)\theta}{(1-\nu\theta)}$, where D_0 is a positive constant and $0\le\nu\le1$, to model the displacement of one fluid by another in a porous medium in the absence of gravity.

This solution was subsequently extended by Rogers et al. [14] to include the effect of gravitational gradients. Their model is in the form

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x} \right] - V(t)\frac{\partial f(\theta)}{\partial x} - \frac{\partial}{\partial x}[K(\theta)(1 - f(\theta)],$$

where $K(\theta)$, the hydraulic conductivity was taken as $K(\theta) = K_{sat}\theta$, with K_{sat} being the saturated conductivity.

In 1988, Broadbridge and White [15], presented analytic solutions for a nonlinear Fokker-Planck diffusion-convection model describing constant rate rainfall infiltration in uniform soils and other porous materials, which has the form

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x} \right] - K'(\theta)\frac{\partial \theta}{\partial x},$$

where t is the time, x is the depth (positive downwards), $\theta(x,t)$ is the volumetric soil water content, $D(\theta)$ is the soil water diffusivity, $K(\theta)$ is the hydraulic conductivity and $K'(\theta) = \frac{dK}{d\theta}$. The model is based on the Darcy-Buckingham approach to unsaturated water flow. It assumes simple functional forms for the soil water diffusivity $D(\theta)$ and hydraulic conductivity $K(\theta)$, which are $D(\theta) = \frac{a}{(b-\theta)^2}$ and $K(\theta) = \beta + \gamma(b-\theta) + \frac{\lambda}{(2(b-\theta))}$. Their model describes the dependence of the hydraulic conductivity on water content. The analytic solutions describe the temporal development of the water content profile during rainfall.

They predict the time dependence of both surface moisture content and surface soil water potential and the shape of large-time.

In 1993, Sander et al [16] derived a new exact solution to the nonlinear diffusion-convection equation for two-phase flow in porous media. The solution is sought for Dirichlet boundary conditions and a diffusivity of the form $D(\theta) = \frac{D_0}{(1-\nu\theta)^2}$.

The functional form of the fractional flow $f(\theta)$ is then found from a condition which determines the integrability of the resulting nonlinear ordinary differential equation. They found that, $f(\theta)$ has the form $f(\theta) = \frac{(1-\nu+\nu C_2)\theta - \nu C_2\theta^2}{(1-\nu\theta)}$.

For $C_2 = 0$, this case of the solution of Fokas and Yortsos [1] is extended to satisfy the Dirichlet boundary conditions.

They considered the physical application of the solution for air and water flow to in-situ agricultural fields. They take into consideration the effect of sorptivity, which measure the strength of the capillarity forces acting on the flow process.

They also derive some asymptotic expansions in the limits of the parameters ν and θ_s approaching either one or zero.

2 Mathematical Formulation of the Problem

Consider the nonlinear diffusion-convection equation for two- phase flow in porous media in the form

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x} \right] - V(t)\frac{\partial f(\theta)}{\partial x}, \tag{1}$$

with the boundary conditions

(i) $\theta(x,t) = \theta_S$, at $x = 0$, $t > 0$, (2)

(ii) $\theta(x,t) = 0$, at $x = \infty$, $t > 0$, (3)

and initial condition

$\theta(x,t) = 0$, at $t = 0$, $x > 0$. (4)

where t is the time, x is the space variable, θ is the normalized volumetric water content ($0 \le \theta \le 1$), defined as $\theta = \frac{\Theta - \Theta_r}{\Theta_{Sat} - \Theta_r}$, where Θ, Θ_r and Θ_{Sat} are the water content, residual and saturated water contents respectively, $D(\theta)$ is the water diffusivity, $f(\theta)$ is the fractional flow function which is a monotonically increasing function of θ with $0 \le f \le 1$ $\left(\frac{d f}{d\theta} \ge 0\ ,\ f(0) = 0 \textit{ and } f(1) = 1\right)$ which express the effect of both the viscosity and conductivity of the fluids flowing through the porous medium, θ_S is the normalized surface water content ($0 < \theta_S \le 1$) and $V(t)$ is the combined flow rate of both fluids which is a function of time only, since we assume incompressibility of both fluids.

Assume

$D(\theta) = \frac{D_0}{(1-\nu\theta)^2}$ and $f(\theta) = \frac{(1-\nu)\theta}{(1-\nu\theta)}$, (5)

where D_0 is a positive constant and $0 \le \nu \le 1$.

Substitution from (5) into (1) yields

$$\frac{\partial\theta}{\partial t} = \frac{D_0}{(1-\nu\theta)^2}\frac{\partial^2\theta}{\partial x^2} + \frac{2\nu D_0}{(1-\nu\theta)^3}\left(\frac{\partial\theta}{\partial x}\right)^2 - V(t)\frac{(1-\nu)}{(1-\nu\theta)^2}\frac{\partial\theta}{\partial x}, \quad (6)$$

which can be written as

$$(1-\nu\theta)^3\theta_t - (1-\nu\theta)D_0\theta_{xx} - 2\nu D_0(\theta_x)^2 + (1-\nu)(1-\nu\theta)V(t)\theta_x = 0\,, \quad (7)$$

where, subscripts denote partial derivatives with respect to the time t and space x.

3 Solution of the problem

We first determine a one-parameter Lie group of transformations, under which (2.7) is invariant, then we use this group to determine similarity variables.

3.1 Lie point symmetries

We seek point symmetries of the form

$$\bar{x} = x + \varepsilon\phi(x,t,\theta) + O(\varepsilon^2)\,,$$

$$\bar{t} = t + \varepsilon\zeta(x,t,\theta) + O(\varepsilon^2),$$
$$\bar{\theta} = \theta + \varepsilon\eta(x,t,\theta) + O(\varepsilon^2). \tag{8}$$
where "ε" is a small parameter.

Each one-parameter Lie group of point transformations is obtained by exponentiating its infinitesimal generator, which is

$$X \equiv \phi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial \theta}. \tag{9}$$

Equivalently, we can obtain $(\bar{x},\bar{t},\bar{\theta})$ by solving

$$\frac{d\bar{x}}{d\varepsilon} = \phi(\bar{x},\bar{t},\bar{\theta}), \quad \frac{d\bar{t}}{d\varepsilon} = \zeta(\bar{x},\bar{t},\bar{\theta}), \quad \frac{d\bar{\theta}}{d\varepsilon} = \eta(\bar{x},\bar{t},\bar{\theta}), \tag{10}$$

subjected to the initial conditions

$$(\bar{x},\bar{t},\bar{\theta})\big|_{\varepsilon=0} \equiv (x,t,\theta). \tag{11}$$

The solution $\theta = \theta(x,t)$, is mapped to itself by the group of transformations generated by "X" if

$$\Phi_\theta \equiv X[\theta - \theta(x,t)] = 0 \text{ when } \theta = \theta(x,t), \tag{12}$$

This condition can be expressed by using the characteristic of the group, which is

$$\Phi_\theta = \eta - \phi\frac{\partial\theta}{\partial x} - \zeta\frac{\partial\theta}{\partial t}. \tag{13}$$

From (12) and (13), the solution $\theta = \theta(x,t)$, is invariant provided that

$$\Phi_\theta = 0 \text{ when } \theta = \theta(x,t), \tag{14}$$

from which, (14) can be written as,

$$\phi\frac{\partial\theta}{\partial x} + \zeta\frac{\partial\theta}{\partial t} = \eta. \tag{15}$$

Equation (15) is called the invariant surface condition, which is a quasilinear equation. The subsidiary equations may be written as

$$\frac{dx}{\phi(x,t,\theta)} = \frac{dt}{\zeta(x,t,\theta)} = \frac{d\theta}{\eta(x,t,\theta)}. \tag{16}$$

To calculate the prolongation of a given transformation, we need to differentiate (8) with respect to each of the parameters, x and t. To do this, we introduce the following total derivatives:

$$D_x \equiv \partial_x + \theta_x\partial_\theta + \theta_{xx}\partial_{\theta_x} + \theta_{xt}\partial_{\theta_t} + \dots\dots\dots,$$

$$D_t \equiv \partial_t + \theta_t\partial_\theta + \theta_{xt}\partial_{\theta_x} + \theta_{tt}\partial_{\theta_t} + \dots\dots \tag{17}$$

The prolongation of the point transformation (8), to first and second derivatives is

$$\bar\theta_{\bar x} = \theta_x + \varepsilon\,\eta^x(x,t,\theta,\theta_x,\theta_t) + O(\varepsilon^2),$$

$$\bar\theta_{\bar t} = \theta_t + \varepsilon\,\eta^t(x,t,\theta,\theta_x,\theta_t) + O(\varepsilon^2),$$

$$\bar\theta_{\bar x\bar x} = \theta_{xx} + \varepsilon\,\eta^{xx}(x,t,\theta,\theta_x,\theta_t,\theta_{xt},\theta_{xx}) + O(\varepsilon^2). \tag{18}$$

Where,

$$\eta^x(x,t,\theta,\theta_x,\theta_t) = D_x\eta - \theta_x D_x\phi - \theta_t D_x\zeta,$$
$$\eta^t(x,t,\theta,\theta_x,\theta_t) = D_t\eta - \theta_x D_t\phi - \theta_t D_t\zeta,$$
$$\eta^{xx}(x,t,\theta,\theta_x,\theta_t,\theta_{xt},\theta_{xx}) = D_x\eta^x - \theta_{xx} D_x\phi - \theta_{xt} D_x\zeta. \tag{19}$$

A vector X given by (9), is said to be a Lie point symmetry vector field for (7) if

$$X^{[2]}\left((1-\nu\theta)^3\theta_t - (1-\nu\theta)\,D_0\,\theta_{xx} - 2\nu\,D_0(\theta_x)^2 + (1-\nu)(1-\nu\theta)V(t)\,\theta_x\right) = 0, \tag{20}$$

where,

$$X^{[2]} \equiv \phi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial\theta} + \eta^x\frac{\partial}{\partial\theta_x} + \eta^t\frac{\partial}{\partial\theta_t} + \eta^{xx}\frac{\partial}{\partial\theta_{xx}}. \tag{21}$$

Then, equation (20) can be written as

$$(1-\nu)(1-\nu\theta)\,V'(t)\,\zeta\,\theta_x + \left(\nu D_0\theta_{xx} - 3\nu(1-\nu\theta)^2\theta_t - \nu(1-\nu)V(t)\,\theta_x\right)\eta + \left((1-\nu)(1-\nu\theta)V(t) - 4\nu D_0\theta_x\right)\eta^x + (1-\nu\theta)^3\eta^t - (1-\nu\theta)\,D_0\eta^{xx} = 0. \tag{22}$$

Substitution from (19) into (22), leads to a large expression, then, by equating to zero the coefficients of θ_{xt} and $\theta_x\theta_{xt}$, we obtain after simplification

$$\zeta_x = 0, \tag{23}$$
$$\zeta_\theta = 0. \tag{24}$$

From (23) and (24), we get

$$\zeta = \zeta(t). \tag{25}$$

Substitution from (25) into (22), removes many terms. By equating to zero the coefficients of $\theta_x, (\theta_x)^2, (\theta_x)^3, \theta_{xx}, \theta_x\theta_{xx}$ and remaining terms, we get

$$(1-\nu)(1-\nu\theta)\,V'(t)\,\zeta + 2\nu(1-\nu)\,V(t)\,\eta - 4\nu D_0\eta_x - (1-\nu)(1-\nu\theta)\,V(t)\,\phi_x - (1-\nu\theta)^3\phi_t + (1-\nu)(1-\nu\theta)V(t)\,\zeta_t - 2(1-\nu\theta)\,D_0\,\eta_{x\theta} + (1-\nu\theta)\,D_0\,\phi_{xx} = 0, \tag{26}$$

$$4\nu D_0\phi_x - 2\nu D_0\eta_\theta - 2\nu D_0\zeta_t - (1-\nu\theta) D_0\eta_{\theta\theta} +$$
$$2(1-\nu\theta) D_0\phi_{x\theta} - \frac{6 D_0 \nu^2 \eta}{(1-\nu\theta)} = 0, \tag{27}$$

$$(1-\nu\theta) D_0\phi_{\theta\theta} + 2\nu D_0\phi_\theta = 0, \tag{28}$$

$$2 D_0 \nu\eta = (1-\nu\theta)(2\phi_x - \zeta_t) D_0, \tag{29}$$

$$\phi_\theta = 0, \tag{30}$$

$$D_0\eta_{xx} = (1-\nu\theta)^2 \eta_t + (1-\nu) V(t)\eta_x. \tag{31}$$

Solving the determining equations yields

$$\phi = C_1 x + C_2, \tag{32}$$

$$\zeta = (2C_1 - 2\nu C_3) t + C_4, \tag{33}$$

$$\eta = (1-\nu\theta) C_3, \tag{34}$$

$$V(t) = \frac{C_5}{[(2C_1 - 2\nu C_3)\, t + C_4\,]^{\frac{C_1}{2C_1 - 2\nu C_3}}}. \tag{35}$$

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants.

Thus, the symmetry Lie algebra of (7) is four-dimensional and is generated by the operators

$$X_1 \equiv \frac{\partial}{\partial x}, \quad X_2 \equiv \frac{\partial}{\partial t}, \quad X_3 \equiv x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad X_4 \equiv -2\nu t\frac{\partial}{\partial t} + (1-\nu\theta)\frac{\partial}{\partial\theta}. \tag{36}$$

3.2 Solution invariant under X_3

The general solution of the invariant surface condition (16) is

$$\alpha = \frac{x}{\sqrt{t}}, \quad \theta(x,t) = g(\alpha). \tag{37}$$

Substitution from (37) into (7), yields

$$-\frac{\alpha}{2t}(1-\nu g)^3 \frac{dg}{d\alpha} - \frac{(1-\nu g) D_0}{t}\frac{d^2 g}{d\alpha^2} - \frac{2\nu D_0}{t}\left(\frac{dg}{d\alpha}\right)^2 +$$
$$\frac{(1-\nu)(1-\nu g)}{\sqrt{t}} \frac{C_5}{\left((2C_1 - 2\nu C_3)t + C_4\right)^{\frac{C_1}{2C_1 - 2\nu C_3}}} \frac{dg}{d\alpha} = 0. \tag{38}$$

For (38) to be reduced to an ordinary differential equation in one variable "α", it is necessary that the coefficients should be either constants or functions of "α" only. Thus,

$$C_3 = C_4 = 0. \tag{39}$$

Substitution from (39) into (32)-(35), yields

$$\phi = C_1 x + C_2, \tag{40}$$

$$\zeta = 2C_1 t, \tag{41}$$

$$\eta = 0, \tag{42}$$

$$V(t) = \frac{C_5}{\sqrt{2C_1}\sqrt{t}}. \tag{43}$$

So, the symmetry Lie algebra of (7) is only two-dimensional and is generated by the operators

$$X_1 \equiv \frac{\partial}{\partial x}, \quad X_2 \equiv x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}. \tag{44}$$

The one-parameter group generated by X_1 consists of translation, whereas the symmetry X_2 generates scaling.

The finite transformations corresponding to the symmetries X_1 and X_2 are

$$X_1: \ \bar{x} = x + \varepsilon_1, \quad \bar{t} = t, \qquad \bar{\theta} = \theta \qquad , \tag{45}$$

$$X_2: \ \bar{x} = e^{\varepsilon_2} x, \quad \bar{t} = e^{2\varepsilon_2} t, \quad \bar{\theta} = \theta. \tag{46}$$

where ε_1 and ε_2 are group parameters.

Substituting from (39) into (38), we get

$$(1-\nu g)\frac{d^2 g}{d\alpha^2} + \frac{\alpha}{2D_0}(1-\nu g)^3\frac{dg}{d\alpha} + 2\nu\left(\frac{dg}{d\alpha}\right)^2 - \frac{C_5(1-\nu)(1-\nu g)}{\sqrt{2C_1}\,D_0}\frac{dg}{d\alpha} = 0. \tag{47}$$

Under the similarity variable "α", the initial and boundary conditions (2)-(4) are

$$g(0) = \theta_S, \tag{48}$$

$$g(\infty) = 0. \tag{49}$$

4 Numerical Solution

4.1 Study of the effect of "ν"

Consider $D_0 = 5$ and $\theta_S = 0.95$.

The results for different values of "ν" are plotted in figure (1).

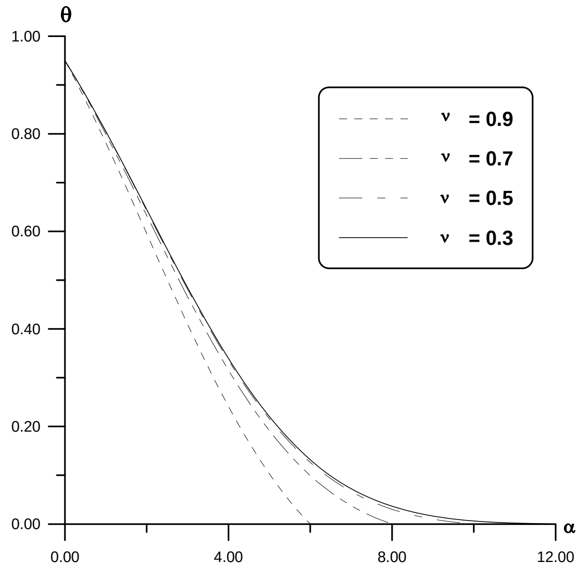

Fig.(1): Effect of "ν" on the normalized volumetric water content $\theta(x,t)$ for
$\mathbf{D_0 = 5}$ and $\theta_S = \mathbf{0.95}$.

4.2 Study of the effect of "θ_S"

Consider $D_0 = 5$ and $\nu = 0.9$.
The results for different values of "θ_S" are plotted in figure (2).

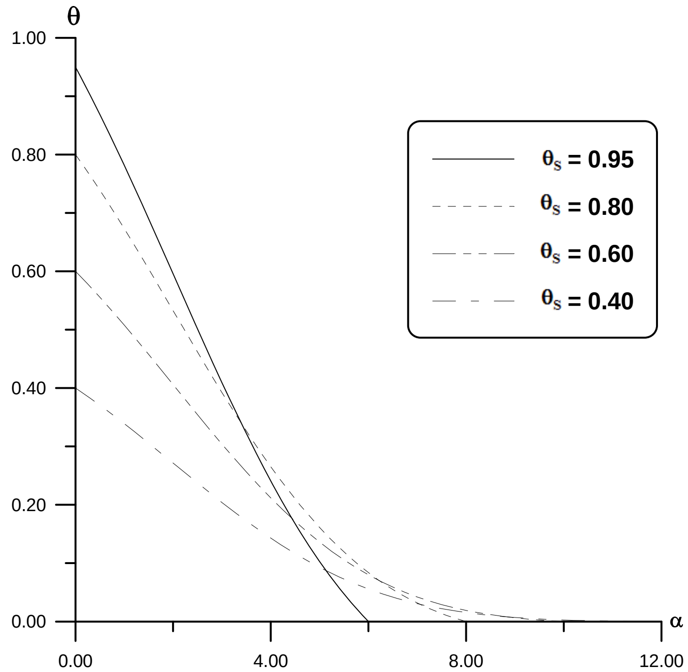

Fig.(2): Effect of "θ_S" on the normalized volumetric water content $\theta(x,t)$ for
$\mathbf{D_0 = 5}$ and $\nu = \mathbf{0.9}$.

4.3 Study of the effect of "D_0"

Consider $\nu = 0.9$ and $\theta_S = 0.95$.

The results for different values of "D_0" are plotted in figure (3).

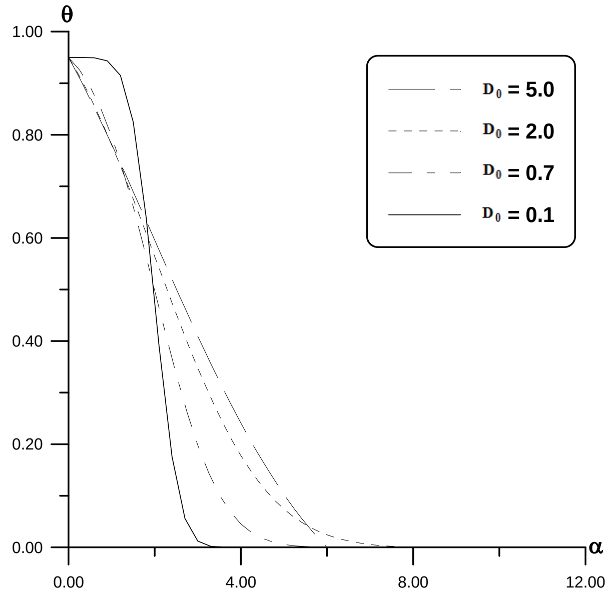

Fig. (3): Effect of "D_0" on the normalized volumetric water content $\theta(x,t)$ for $\nu = 0.9$ and $\theta_S = 0.95$.

5 Results and discussion

Lie-group method is applicable to both linear and non-linear partial differential equations, which leads to similarity variables that may be used to reduce the number of independent variables in partial differential equations. By determining the transformation group under which a given partial differential equation is invariant, we can obtain information about the invariants and symmetries of that equation. This information can be used to determine similarity variables that will reduce the number of independent variables in the system. In this work, we have used Lie symmetry techniques to obtain similarity reductions of a nonlinear diffusion-convection equation arising in modeling the immiscible flow of two fluids in a porous medium, which was presented by Fokas and Yortsos [1], for oil and water flow in petroleum-reservoir, and is given by (2.1) with the assumptions $D(\theta) = \frac{D_0}{(1-\nu\theta)^2}$ and $f(\theta) = \frac{(1-\nu)\theta}{(1-\nu\theta)}$.

By determining the transformation group under which a given partial differential equation is invariant, we obtained information about the invariants and symmetries of that equation. This information, in turn, was used to determine similarity variables that reduced the number of independent variables by one. The ordinary differential equation (47) with the appropriate corresponding condition (48) and (49) is solved numerically using shooting method technique.

We study the effect of the different parameters on the volumetric water content $\theta(x,t)$. The effect of "ν" on the volumetric water content $\theta(x,t)$ increases as "ν" decreases, figure (1). The effect of "θ_S" on the volumetric water content $\theta(x,t)$ increases with the increase of "θ_S", figure (2). The effect of "D_0" on the volumetric water content $\theta(x,t)$ increases as "D_0" increases, figure (3).

Comparing this work with the study of Sander et.al [16], we fond that, the volumetric water content profiles have a similar trend to that obtained by them

References

[1] A.S. Fokas, Y.C. Yortsos., On the exactly solvable equation $S_t = \left((\beta S+\gamma)^{-2} S_x\right)_x + \alpha(\beta S+\gamma)^{-2} S_x$ occurring in two-phase flow in porous media, SIAM J. Appl. Math., 42(1982) 318-332.

[2] Z. Yi, M. Fengxiang, Lie symmetries of mechanical systems with unilateral holonomic constraints, Chinese Science Bulletin, 45 (2000) 1354-1358.

[3] B. Moritz, W. Schwalm, D. Uherka, Finding Lie groups that reduce the order of discrete dynamical systems, J.Phys.A: Math. 31 (1998) 7379-7402.

[4] M.C. Nucci, P.A. Clarkson, The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh-Nagumo equation, Physics Letters A, 164 (1992) 49-56.

[5] P. Basarab, V. Lahno, Group classification of nonlinear partial differential equations: a new approach to resolving the problem, Proceedings of Institute of Mathematics of NAS of Ukraine, 43 (2002) 86-92.

[6] G.I. Burde, Expanded Lie group transformations and similarity reductions of differential equations, Proceedings of Institute of Mathematics of NAS of Ukraine, 43 (2002) 93-101.

[7] I.M. Tsyfra, Conditional symmetry reduction and invariant solutions of nonlinear wave equations, Proceedings of Institute of Mathematics of NAS of Ukraine, 43 (2002) 229-233.

[8] M.L. Gandarias, M.S. Bruzon,, Classical and nonclassical symmetries of a generalized Boussinesq equation, J.of Nonlinear Mathematical Physics, 5 (1998) 8-12.

[9] N.H. Ibragimov, Elementary Lie group analysis and ordinary differential equations, Wiley, New York, 1999.

[10] P.E. Hydon, Symmetry methods for differential equations, CUP, Cambridge, 2000.

[11] J.M. Hill, Solution of differential equations by means of one-parameter groups, Pitman Publishing Co., 1982.

[12] R. Seshadri, T.Y. Na, Group invariance in engineering boundary value problems, Springer-Verlag, New York, 1985.

[13] M. Awad, I. Turner, Flux-limiting and non-linear solution techniques for simulation of transport in porous media, ANZIAM J. 42(2000) C157-C182.

[14] C. Rogers, M.P. Stallybrass, D.L. Clements, On two phase filtration under gravity and with boundary infiltration: Application of a Backlund transformation, Nonlinear Anal. Theory Methods Appl., 7(1983) 785-799.

[15] P. Broadbridge, I. White, Constant rate rainfall infiltration: A Versatile nonlinear model 1. Analytic solution, Water Resour. Res. 24(1988) 145-154.

[16] G.C. Sander, J. Norbury, S.W. Weeks, An exact solution to the nonlinear diffusion-convection equation for two-phase flow, Q.J1 Mech appl. Math. 46(1993) 709-727.

Received: July 4, 2006