Robotics

Lecture 5

Forward Kinematics
Examples

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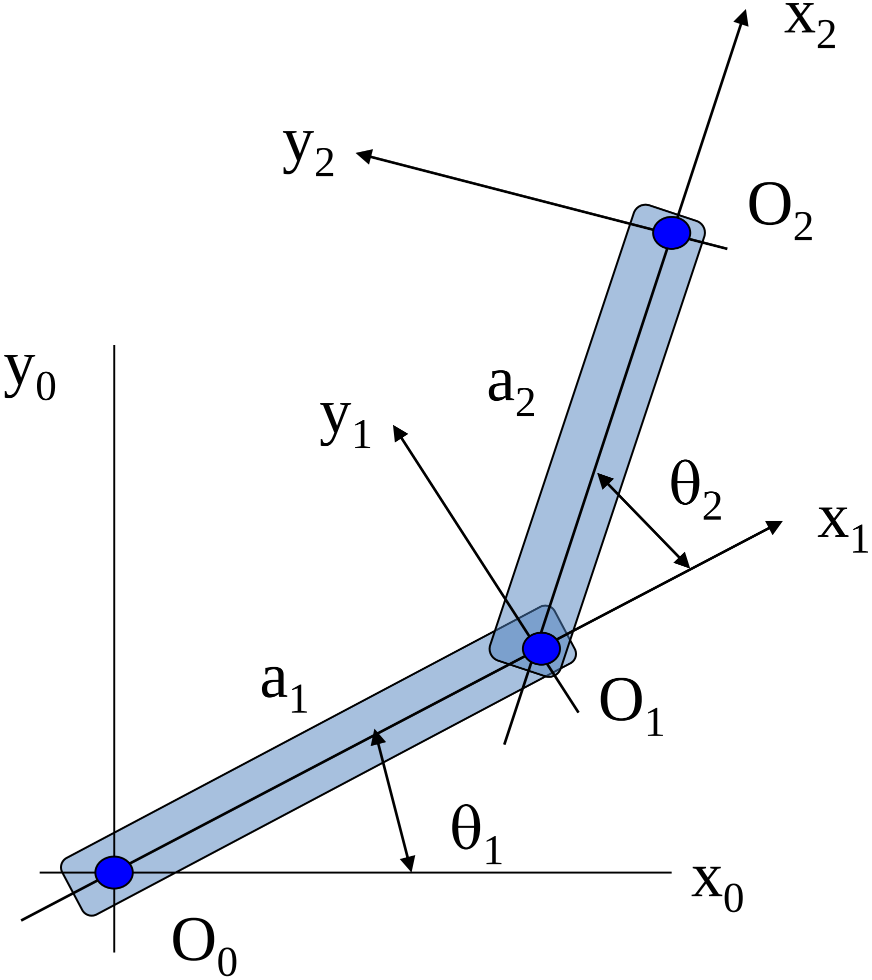
http://www.aast.edu/cv.php?disp_unit=346&ser=68525

Examples

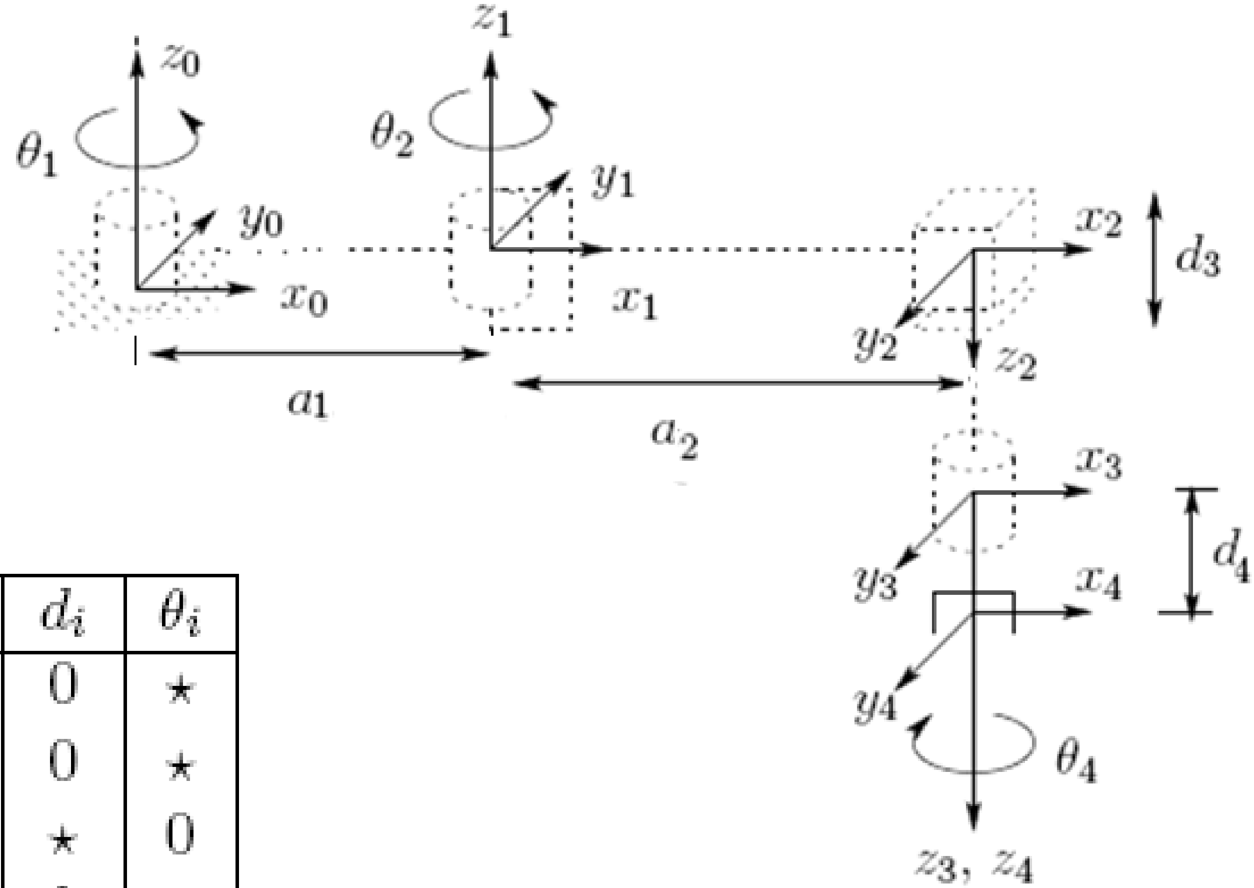
- Base frame O_0
- All Z ‘s are normal to the page

	distance between the Z_i and Z_{i-1} along the X_i	angle between the Z_i and Z_{i-1}	Distance along Z_{i-1}	angle between the X_{i-1} and X_i measured about Z_{i-1}
Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable



Example 2



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\star$
2	a_2	180	0	$\star$
3	0	0	$\star$	0
4	0	0	d_4	$\star$

* joint variable

Example 2

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

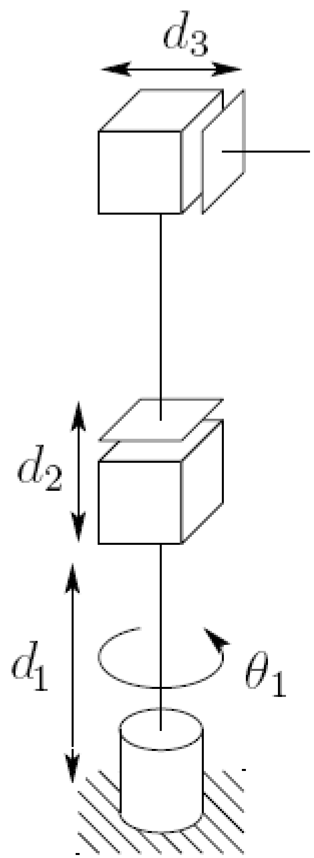
$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

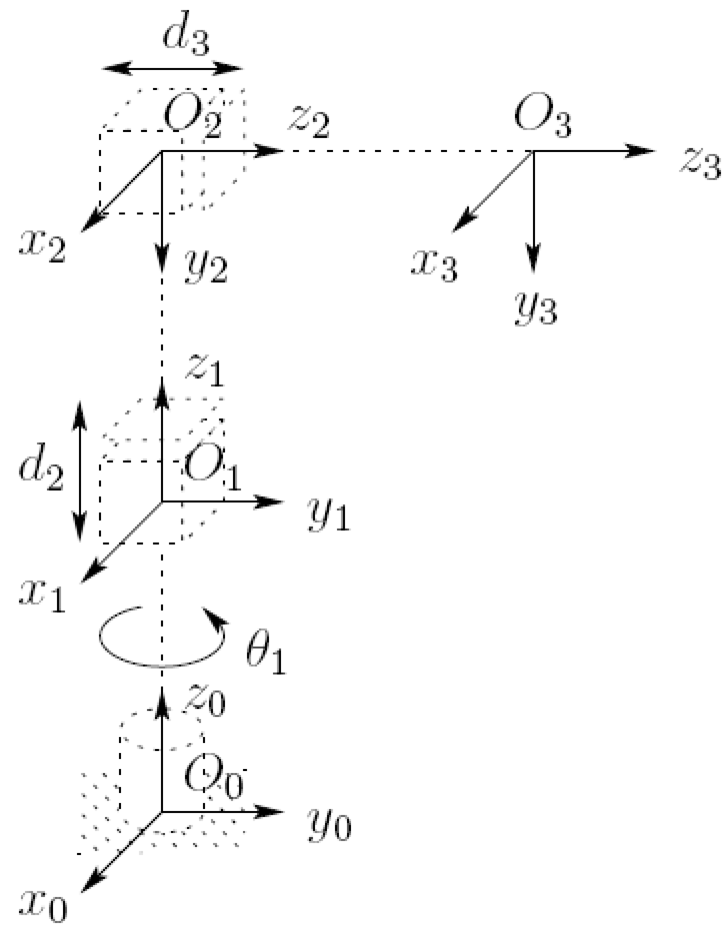
Example 3
The three links cylindrical



Example 3
The three links cylindrical

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

$*$ variable



Example 3
The three links cylindrical

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

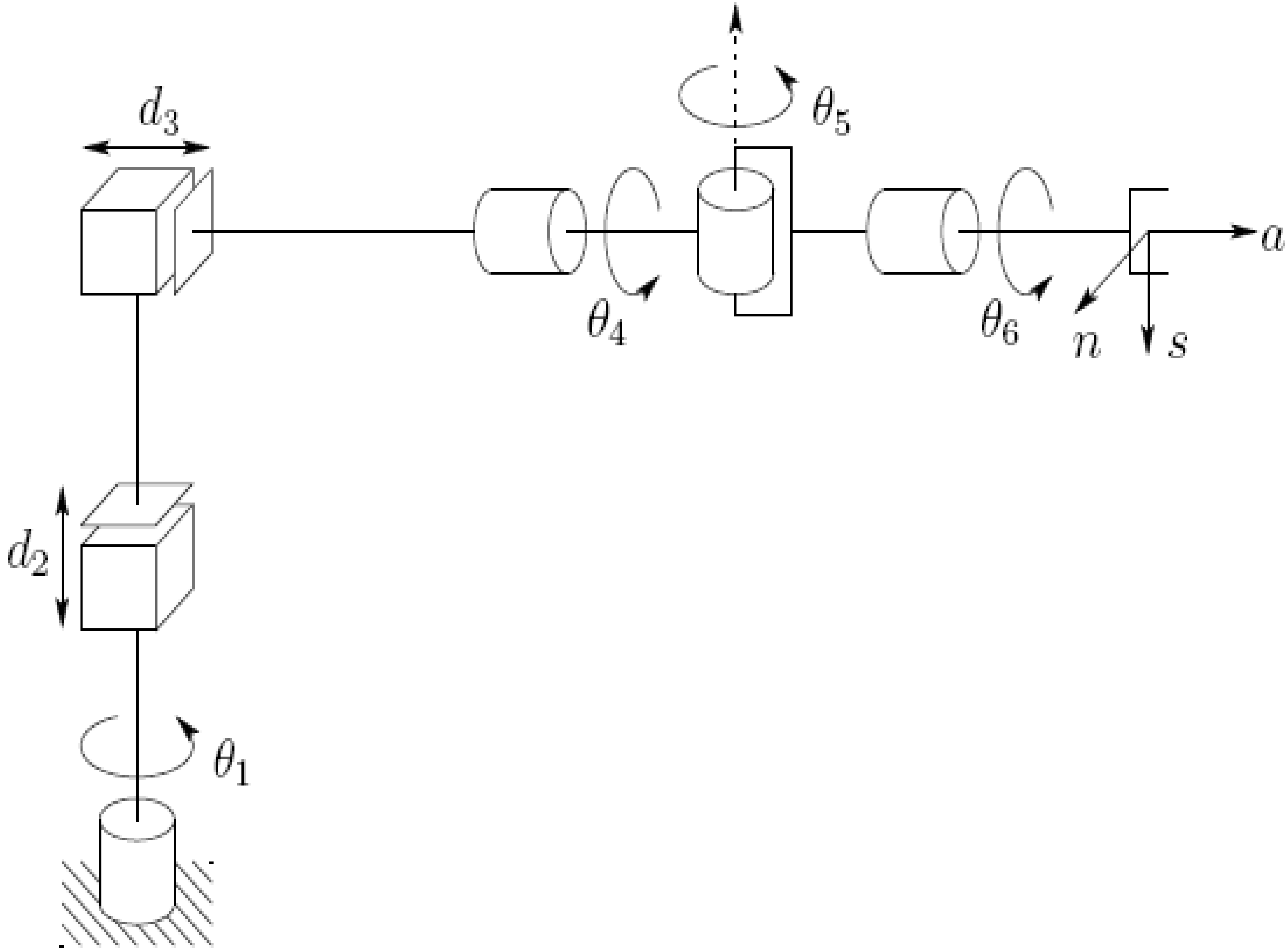
$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

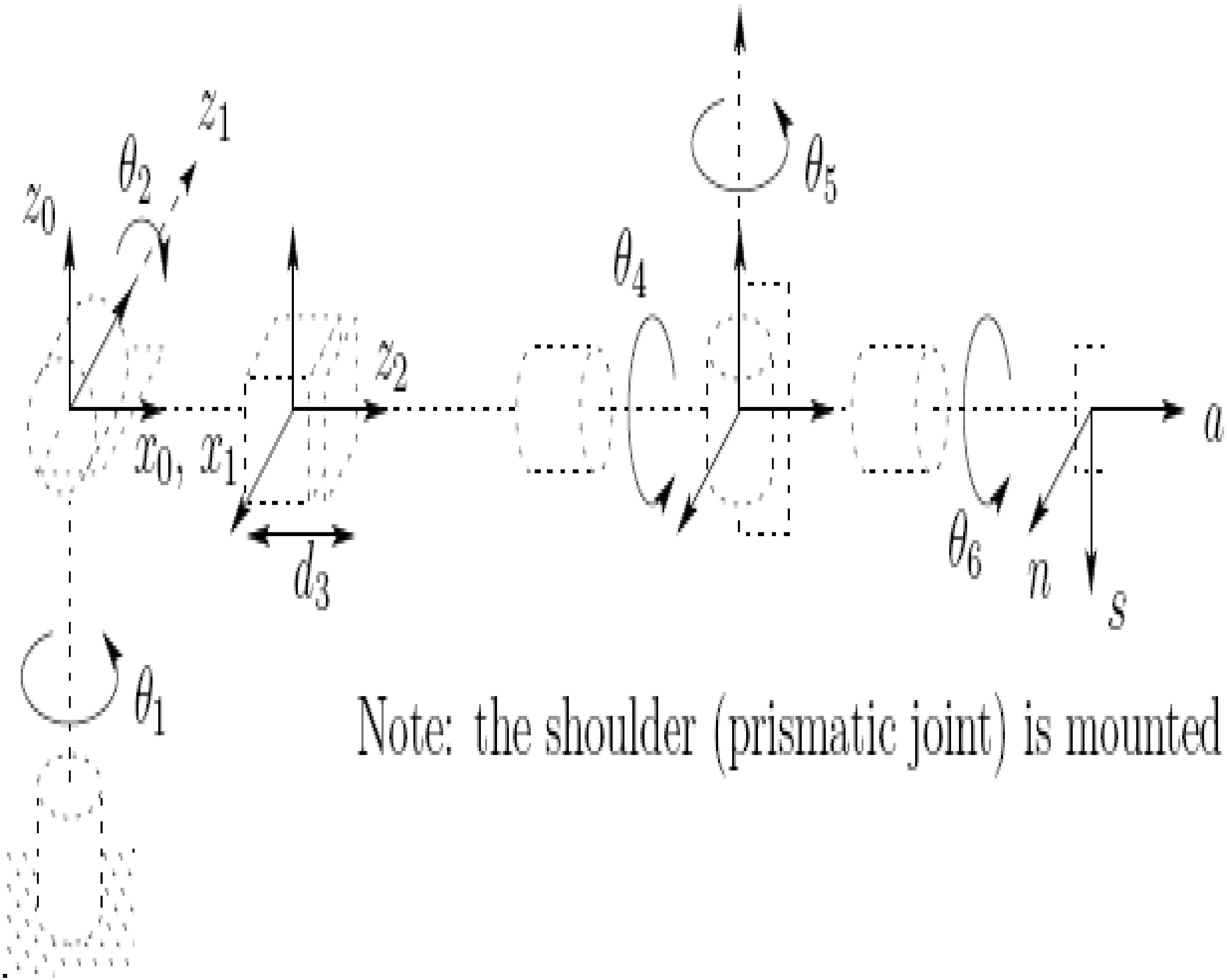
Example 3
The three links cylindrical

$$T_3^0 = A_1 A_2 A_3 \quad = \quad \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples



d_3
θ_5
a
θ_4
θ_6
n
s
d_2
θ_1



Note: the shoulder (prismatic joint) is mounted wrong.

Link	d_i	a_i	α_i	θ_i
1	0	0	-90	$\star$
2	d_2	0	$+90$	$\star$
3	$\star$	0	0	0
4	0	0	-90	$\star$
5	0	0	$+90$	$\star$
6	d_6	0	0	$\star$

* joint variable

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

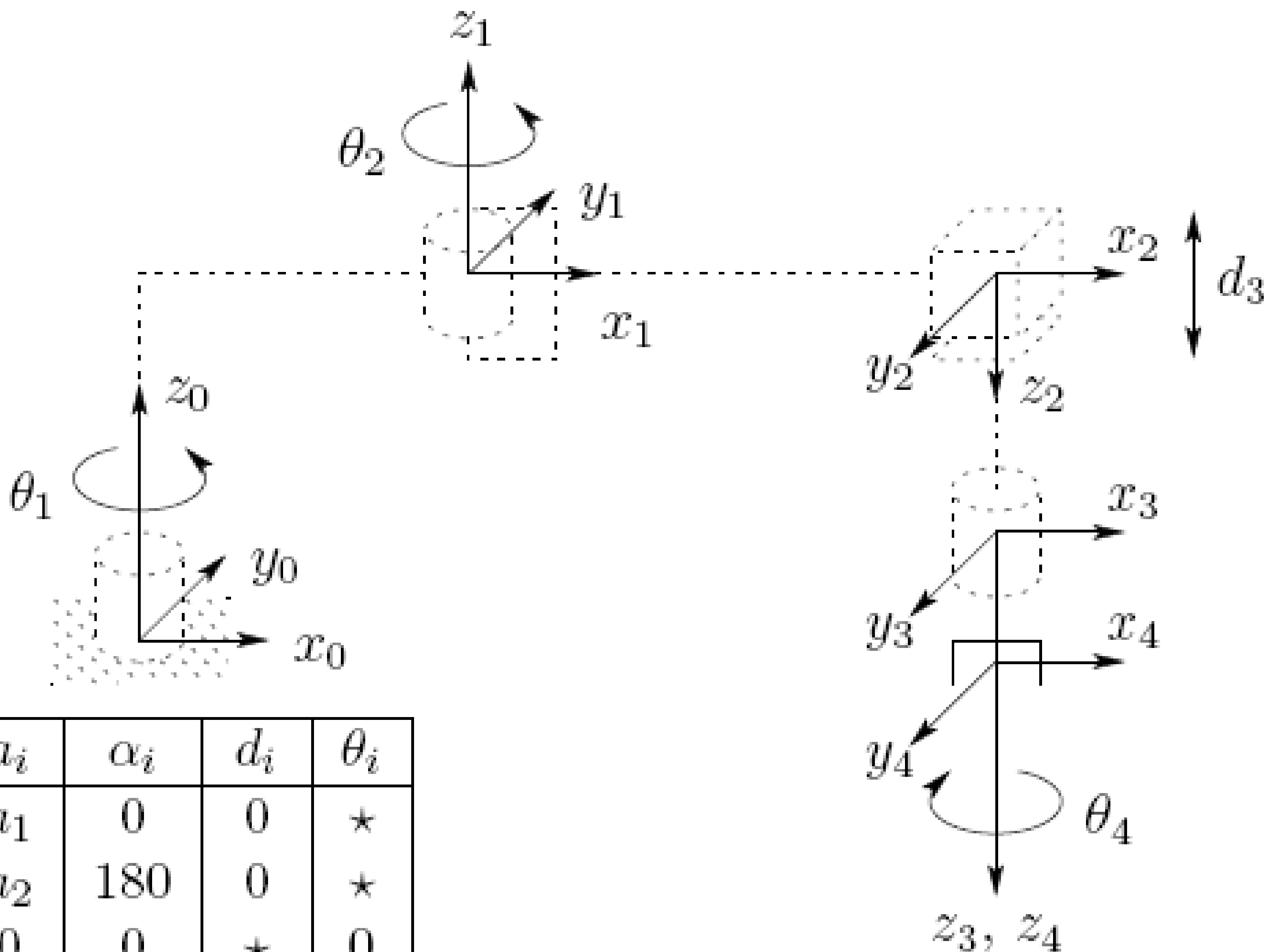
$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} T_6^0 &= A_1 \cdots A_6 \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5).
\end{aligned}
$$



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\star$
2	a_2	180	0	$\star$
3	0	0	$\star$	0
4	0	0	d_4	$\star$

* joint variable

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2 c_2 \\ s_2 & -c_2 & 0 & a_2 s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

End of Lec

Inverse Kinematics (IK)

"Given a goal position find the joint angles for the robot arm"

Inverse Kinematics

- The inverse kinematics is needed in the control of manipulators.
- Solving the inverse kinematics is computationally expansive and generally takes a very long time in the real time control of manipulators.
- IK generally harder than FK
- Sometimes no analytical solution
- Sometimes multiple solutions
- Sometimes no solution
  - Outside workspace

Inverse kinematics

- **Analytical Method** — Joint variables solved according to given configuration data
  - **Geometric solution** — For simple structures,2-DOF
  - **Algebraic solution** — For more links and in 3 dimensions
- **Numerical Method** — Joint variables obtained by numerical techniques

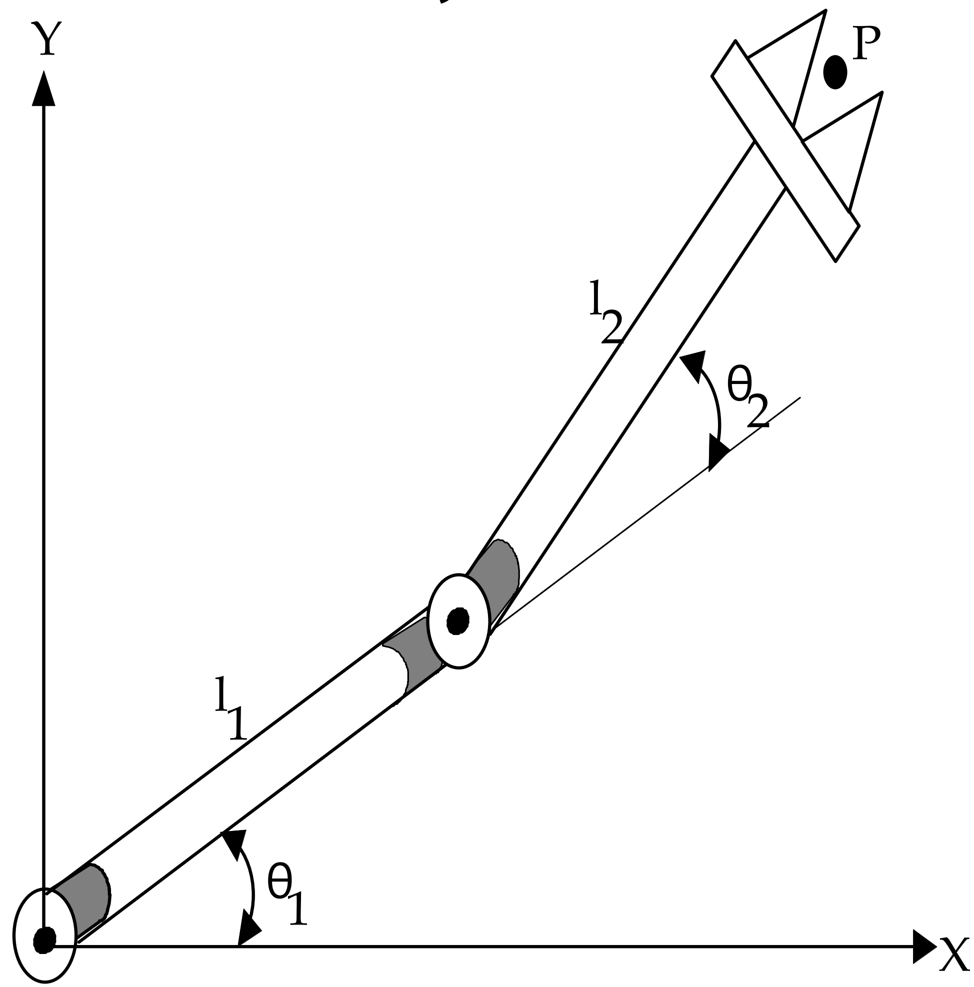
Geometric Solution Approach

Geometric Solution Approach

- It is applied to the simple robot structures, such as, 2-DOF planer manipulator whose joints are both revolute.
- In the shown Figure, the components of point P (p_x , p_y) are determined as follows.

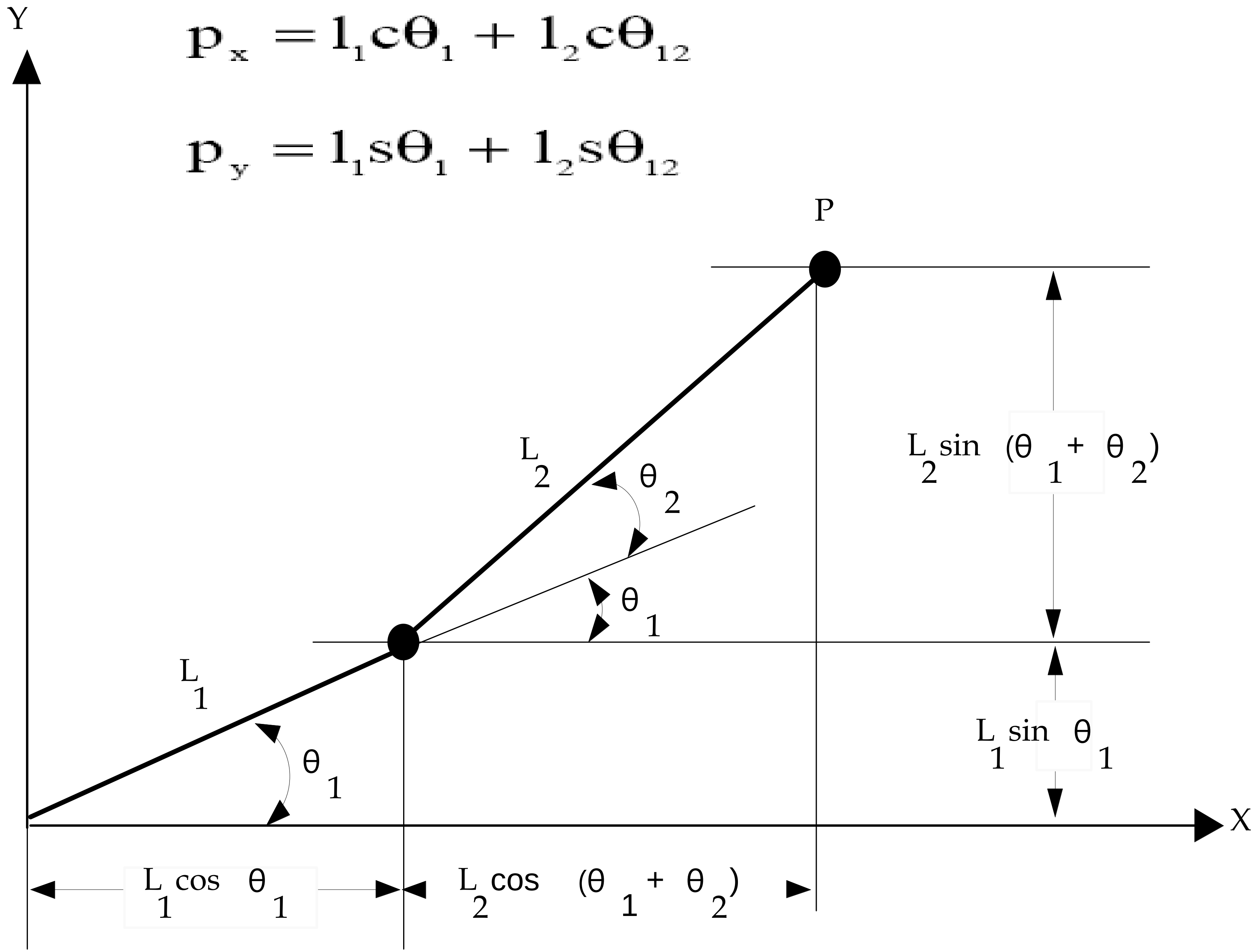
$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$



$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$



$$p_x = l_1 c\theta_1 + l_2 c\theta_{12}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12}$$

- The solution of θ_2 can be computed from summation of squaring both previous equations

$$p_x^2 = l_1^2 c^2\theta_1 + l_2^2 c^2\theta_{12} + 2 l_1 l_2 c\theta_1 c\theta_{12}$$

$$p_y^2 = l_1^2 s^2\theta_1 + l_2^2 s^2\theta_{12} + 2 l_1 l_2 s\theta_1 s\theta_{12}$$

$$p_x^2 + p_y^2 = l_1^2 (c^2\theta_1 + s^2\theta_1) + l_2^2 (c^2\theta_{12} + s^2\theta_{12}) + 2 l_1 l_2 (c\theta_1 c\theta_{12} + s\theta_1 s\theta_{12})$$

- Since $c^2\theta_1 + s^2\theta_1 = 1$, the equation is simplified as:

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 (c\theta_1 [c\theta_1 c\theta_2 - s\theta_1 s\theta_2] + s\theta_1 [s\theta_1 c\theta_2 + c\theta_1 s\theta_2])$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 (c^2\theta_1 c\theta_2 - c\theta_1 s\theta_1 s\theta_2 + s^2\theta_1 c\theta_2 + c\theta_1 s\theta_1 s\theta_2)$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 (c\theta_2 [c^2\theta_1 + s^2\theta_1])$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c\theta_2$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1l_2c\theta_2$$

and so

$$c\theta_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

Since, $c^2\theta_i + s^2\theta_i = 1$ (i =1,2,3,……), $s\theta_2$ is obtained as

$$s\theta_2 = \pm\sqrt{1 - \left(\frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)^2}$$

Finally, two possible solutions for θ_2 can be written as

$$\theta_2 = A\tan 2\left(\pm\sqrt{1 - \left(\frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)^2}, \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)$$

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12} \quad \text{.........1}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12} \quad \text{.........2}$$

- The solution of θ_1 multiply each side of equation 1 by $c\theta_1$ and equation 2 by $s\theta_1$ and add the resulting equations in order to find the solution of θ_1 in terms of link parameters and the known variable θ_2.

$$c\theta_1 p_x = l_1 c^2\theta_1 + l_2 c^2\theta_1 c\theta_2 - l_2 c\theta_1 s\theta_1 s\theta_2$$

$$s\theta_1 p_y = l_1 s^2\theta_1 + l_2 s^2\theta_1 c\theta_2 + l_2 s\theta_1 c\theta_1 s\theta_2$$

$$c\theta_1 p_x + s\theta_1 p_y = l_1 (c^2\theta_1 + s^2\theta_1) + l_2 c\theta_2 (c^2\theta_1 + s^2\theta_1)$$

The simplified equation obtained as follows.

$$c\theta_1 p_x + s\theta_1 p_y = l_1 + l_2 c\theta_2 \quad \text{...........3}$$

$$p_x = l_1 c\theta_1 + l_2 c\theta_{12} \qquad \text{.........1}$$

$$p_y = l_1 s\theta_1 + l_2 s\theta_{12} \qquad \text{.........2}$$

- Multiply each side of equation 1 by $-s\theta_1$ and equation 2 by $c\theta_1$ and add the resulting equations

$$-s\theta_1 p_x = -l_1 s\theta_1 c\theta_1 - l_2 s\theta_1 c\theta_1 c\theta_2 + l_2 s^2\theta_1 s\theta_2$$

$$c\theta_1 p_y = l_1 s\theta_1 c\theta_1 + l_2 c\theta_1 s\theta_1 c\theta_2 + l_2 c^2\theta_1 s\theta_2$$

$$-s\theta_1 p_x + c\theta_1 p_y = l_2 s\theta_2 (c^2\theta_1 + s^2\theta_1)$$

The simplified equation is given by

$$-s\theta_1 p_x + c\theta_1 p_y = l_2 s\theta_2 \qquad \text{............4}$$

Now, multiply each side of equation 3 by p_x and equation 4 by p_y and add the resulting equations in order to obtain $c\theta_1$.

$$c\theta_1 p_x^2 + s\theta_1 p_x p_y = p_x(l_1 + l_2 c\theta_2)$$

$$-s\theta_1 p_x p_y + c\theta_1 p_y^2 = p_y l_2 s\theta_2$$

$$c\theta_1(p_x^2 + p_y^2) = p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2$$

and so

$$c\theta_1 = \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}$$

$$c\theta_1 = \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}$$

$s\theta_1$ is obtained as

$$s\theta_1 = \pm\sqrt{1 - \left(\frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right)^2}$$

As a result, two possible solutions for θ_1 can be written

$$\theta_1 = A\tan 2\left(\pm\sqrt{1 - \left(\frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right)^2}, \frac{p_x(l_1 + l_2 c\theta_2) + p_y l_2 s\theta_2}{p_x^2 + p_y^2}\right)$$

Although the planar manipulator has a very simple structure, as can be seen, its inverse kinematics solution based on geometric approach is very difficult .

Algebraic Solution Approach

End of Lec