Control System I
EE 411
State Space Analysis

Lecture 11

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Course Contents

- State Space (SS) modeling of linear systems
- SS Representation from system Block Diagram
- SS from Differential equation
  - phase variables form
  - Canonical form
  - Parallel
  - Cascade
- State transition matrix and its properties
- Eigen values and Eigen Vectors
- SS solution

State Space Definition

- Steps of control system design
  - Modeling: Equation of motion of the system
  - Analysis: test system behavior
  - Design: design a controller to achieve the required specification
  - Implementation: Build the designed controller
  - Validation and tuning: test the overall sysetm
- In SS :Modeling, analysis and design in time domain

SS-Definition

- In the classical control theory, the system model is represented by a transfer function
- The analysis and control tool is based on classical methods such as root locus and Bode plot
- It is restricted to single input/single output system
- It depends only the information of input and output and it does not use any knowledge of the interior structure of the plant,
- It allows only limited control of the closed-loop behavior using feedback control is used

- Modern control theory solves many of the limitations by using a much “richer” description of the plant dynamics.
- The so-called state-space description provide the dynamics as a set of coupled first-order differential equations in a set of internal variables known as state variables, together with a set of algebraic equations that combine the state variables into physical output variables.

SS-Definition

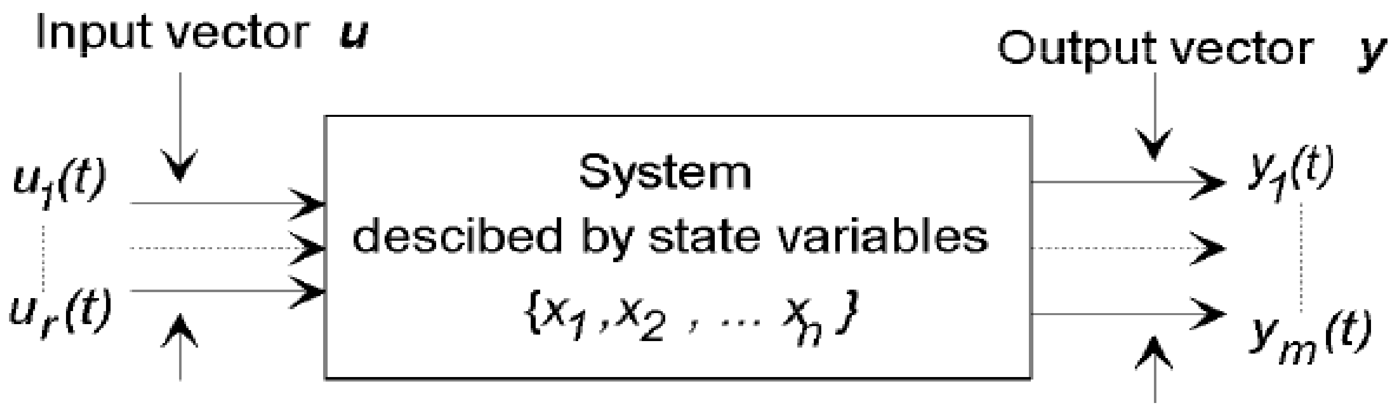
- The Philosophy of SS based on transforming the equation of motions of order *n* (highest derivative order) into an *n* equation of 1st order
- State variable represents storage element in the system which leads to derivative equation between its input and output; it could be a physical or mathematical variables
- # of state=#of storage elements=order of the system
- For example if a system is represented by

$$\frac{d^3y}{dt^3} + 7\frac{d^2y}{dt^2} + 19\frac{dy}{dt} + 13y = 13\frac{du}{dt} + 26u$$

- This system of order 3 then it has 3 state and 3 storage elements

SS-Definition

- The concept of the state of a dynamic system refers to a minimum set of variables, known as state variables, that fully describe the system and its response to any given set of inputs



The state variables are an *internal* description of the system which completely characterize the system state at any time *t*, and from which any output variables *yi*(*t*) may be computed.

The State Equations

A standard form for the state equations is used throughout system dynamics. In the standard form the mathematical description of the system is expressed as a set of *n* coupled first-order ordinary differential equations, known as the *state equations*,

in which the time derivative of each state variable is expressed in terms of the state variables $x1(t), \ldots, xn(t)$ and the system inputs $u1(t), \ldots, ur(t)$.

$$
\begin{aligned}
\dot{x}_1 &= f_1(\mathbf{x}, \mathbf{u}, t) \\
\dot{x}_2 &= f_2(\mathbf{x}, \mathbf{u}, t) \\
\vdots &= \vdots \\
\dot{x}_n &= f_n(\mathbf{x}, \mathbf{u}, t)
\end{aligned}
$$

It is common to express the state equations in a vector form, in which the set of n state variables is written as a *state vector* $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, and the set of r inputs is written as an input vector $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_r(t)]^T$. Each state variable is a time varying component of the column vector $\mathbf{x}(t)$.

$$\dot{\mathbf{x}} = \mathbf{f}\,(\mathbf{x}, \mathbf{u}, t)\,.$$

where $\mathbf{f}\,(\mathbf{x}, \mathbf{u}, t)$ is a *vector* function with n components $f_i\,(\mathbf{x}, \mathbf{u}, t)$.

In this note we restrict attention primarily to a description of systems that are *linear* and *time-invariant* (LTI), that is systems described by linear differential equations with constant coefficients.

$$\begin{array}{ccccccccccccccc}
\dot{x}_1 & = & a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & + & b_{11}u_1 & + & \ldots & + & b_{1r}u_r \\
\dot{x}_2 & = & a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & + & b_{21}u_1 & + & \ldots & + & b_{2r}u_r \\
\vdots & & \vdots & & & & & & & & & & & & \\
\dot{x}_n & = & a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & + & b_{n1}u_1 & + & \ldots & + & b_{nr}u_r
\end{array}$$

$$
\begin{array}{ccccccccccccccccc}
\dot{x}_1 & = & a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & + & b_{11}u_1 & + & \ldots & + & b_{1r}u_r \\
\dot{x}_2 & = & a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & + & b_{21}u_1 & + & \ldots & + & b_{2r}u_r \\
\vdots & & \vdots & & & & & & & & & & & & \\
\dot{x}_n & = & a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & + & b_{n1}u_1 & + & \ldots & + & b_{nr}u_r
\end{array}
$$

where the coefficients a_{ij} and b_{ij} are constants that describe the system. This set of n equations defines the derivatives of the state variables to be a weighted sum of the state variables and the system inputs.

$$
\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1r} \\ b_{21} & & b_{2r} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}
$$

$$
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}
$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where the state vector **x** is a column vector of length *n*, the input vector **u** is a column vector of length *r*, **A** is an $n \times n$ square matrix of the constant coefficients a_{ij}, and **B** is an $n \times r$ matrix of the coefficients b_{ij} that weight the inputs.

A system ***output*** **is** defined to be any system variable of interest. A description of a physical system in terms of a set of state variables does not necessarily include all of the variables of direct engineering interest.

An important property of the linear state equation description is that all system variables may be represented by a linear combination of the state variables *xi* and the system inputs *ui*.

An arbitrary output variable in a system of order *n* with *r* inputs may be written:

$$y(t) = c_1x_1 + c_2x_2 + \ldots + c_nx_n + d_1u_1 + \ldots + d_ru_r$$

$$\begin{array}{ccccccccccccccc}
y_1 & = & c_{11}x_1 & + & c_{12}x_2 & + & \ldots & + & c_{1n}x_n & + & d_{11}u_1 & + & \ldots & + & d_{1r}u_r \\
y_2 & = & c_{21}x_1 & + & c_{22}x_2 & + & \ldots & + & c_{2n}x_n & + & d_{21}u_1 & + & \ldots & + & d_{2r}u_r \\
\vdots & & \vdots & & & & & & & & & & & & \\
y_m & = & c_{m1}x_1 & + & c_{m2}x_2 & + & \ldots & + & c_{mn}x_n & + & d_{m1}u_1 & + & \ldots & + & d_{mr}u_r
\end{array}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & & & \vdots \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \ldots & d_{1r} \\ d_{21} & & d_{2r} \\ \vdots & & \vdots \\ d_{m1} & \ldots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

$$
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \dots & d_{1r} \\ d_{21} & & d_{2r} \\ \vdots & & \vdots \\ d_{m1} & \dots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}
$$

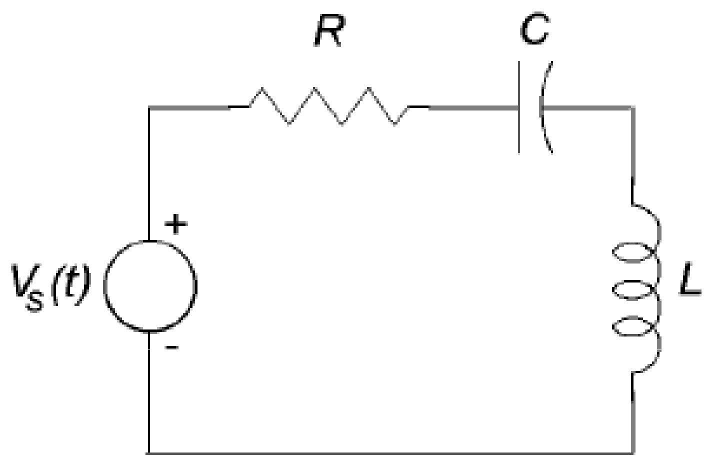
$$
\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}
$$

where **y** is a column vector of the output variables $y_i(t)$, **C** is an $m \times n$ matrix of the constant coefficients c_{ij} that weight the state variables, and D is an $m \times r$ matrix of the constant coefficients d_{ij} that weight the system inputs. For many physical systems the matrix **D** is the null matrix, and the output equation reduces to a simple weighted combination of the state variables:

$$
\mathbf{y} = \mathbf{C}\mathbf{x}.
$$

Example

Find the State equations for the series R-L-C electric circuit shown in



Solution:

capacitor voltage $v_C(t)$ and the inductor current $i_L(t)$ are state variables

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_{in}.$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} V_{in}$$

Prove

Appling KVL on the circuit

$$v_s(t) = R * i + v_c + L\frac{di}{dt} \ldots (1)$$

The relation of capacitor voltage and current $i = c\frac{dv_c}{dt}$

$$then$$

$$\dot{x}_1 = \frac{dv_c}{dt} = \frac{1}{c} i = \frac{1}{c} x_2$$

$$from\ equation\ (1)$$

$$\dot{x}_2 = \frac{di}{dt} = -v_c - R * i + v_s(t)$$

$$\dot{x}_2 = \frac{1}{L}[-x_1 - R * x_2 + u(t)]$$

$$y = v_c = x_1$$

Example

Draw a direct form realization of a block diagram, and write the state equations in phase variable form, for a system with the differential equation

$$\frac{d^3y}{dt^3} + 7\frac{d^2y}{dt^2} + 19\frac{dy}{dt} + 13y = 13\frac{du}{dt} + 26u$$

Solution

we define state variables as

$$x_1 = y,\ x_2 = \dot{y}, and\ x_3 = \ddot{y} + 13u,$$

then the state space representation is

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = x_3 - 13u$$

$$\dot{x}_3 = \dddot{y} - 13\dot{u} = -7\ddot{y} - 19\dot{y} - 13y + 26u$$

$$= -7(x_3 - 13u) - 19x_2 - 13x_1 + 26u$$

$$= -7x_3 - 19x_2 - 13x_1 + 117u$$

$$y = x_1$$

Then the model will be

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -19 & -7 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -13 \\ 117 \end{bmatrix} u(t)$$

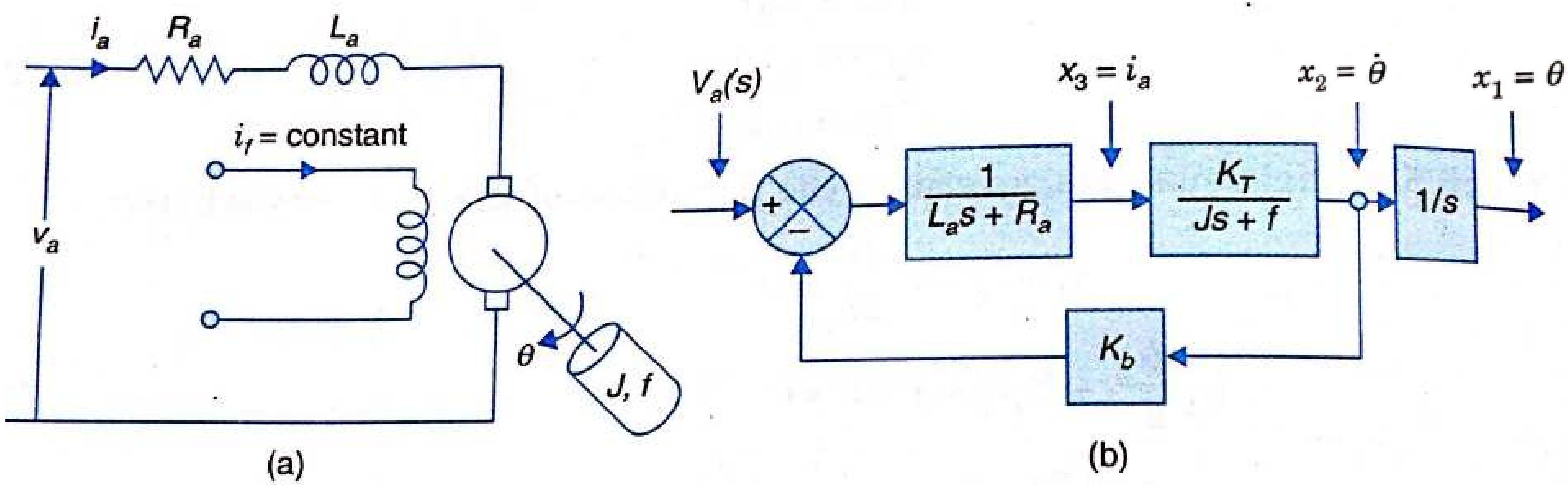
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + [0] u(t)$$

where

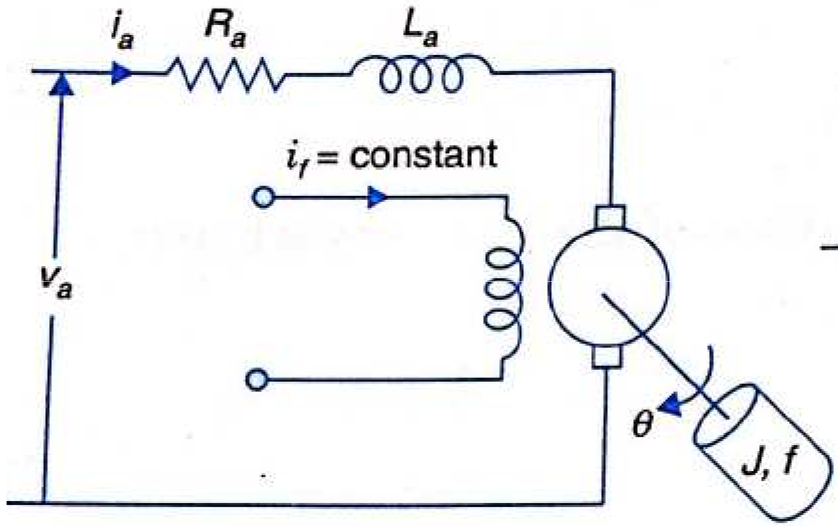
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -19 & -7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -13 \\ 117 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad D = 0$$

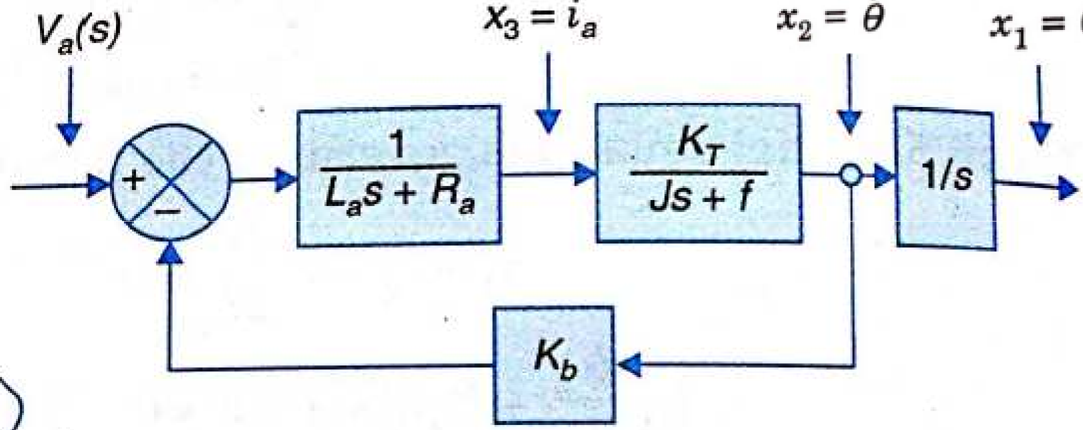
Electro Mechanical System



$$x_1 = \theta\,;\ x_2 = \dot{\theta}\ ;\ \text{and } x_3 = i_a$$



(a)



(b)

$$\frac{1}{s}, \frac{K_T}{Js+f} \text{ and } \frac{1}{L_a s + R_a}$$

$$\dot{x}_1 = x_2$$

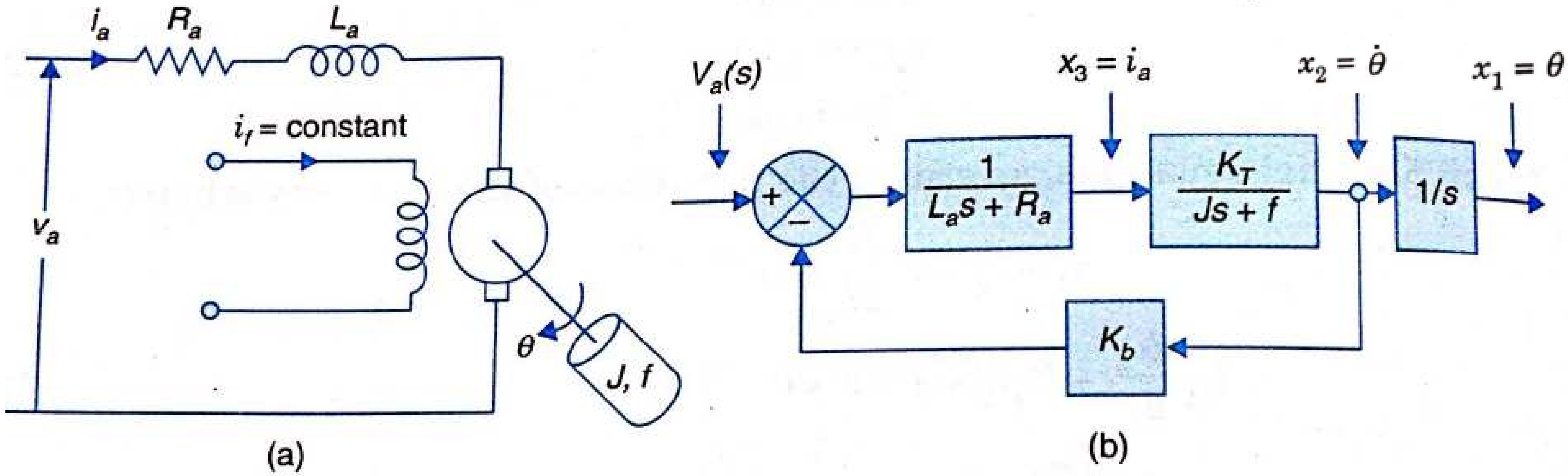
$$J\dot{x}_2 + f x_2 = K_T x_3$$

$$V_a - K_b x_2 = R_a x_3 + L_a \dot{x}_3$$

$$\dot{x}_1 = x_2$$

$$J\dot{x}_2 + fx_2 = K_T x_3$$

$$V_a - K_b x_2 = R_a x_3 + L_a \dot{x}_3$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & K_T/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} v_a \qquad y = \theta = x_1 = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Space Representation

The complete system model for a linear time-invariant system consists of:

(i) a set of *n* state equations, defined in terms of the matrices **A** and **B**, and
(ii) a set of output equations that relate any output variables of interest to the state variables and inputs, and expressed in terms of the **C** and **D** matrices.

The task of modeling the system is to derive the elements of the matrices, and to write the system model in the form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \end{aligned}$$

The matrices **A** and **B** are properties of the system and are determined by the system structure and elements. The output equation matrices **C** and **D** are determined by the particular choice of output variables.

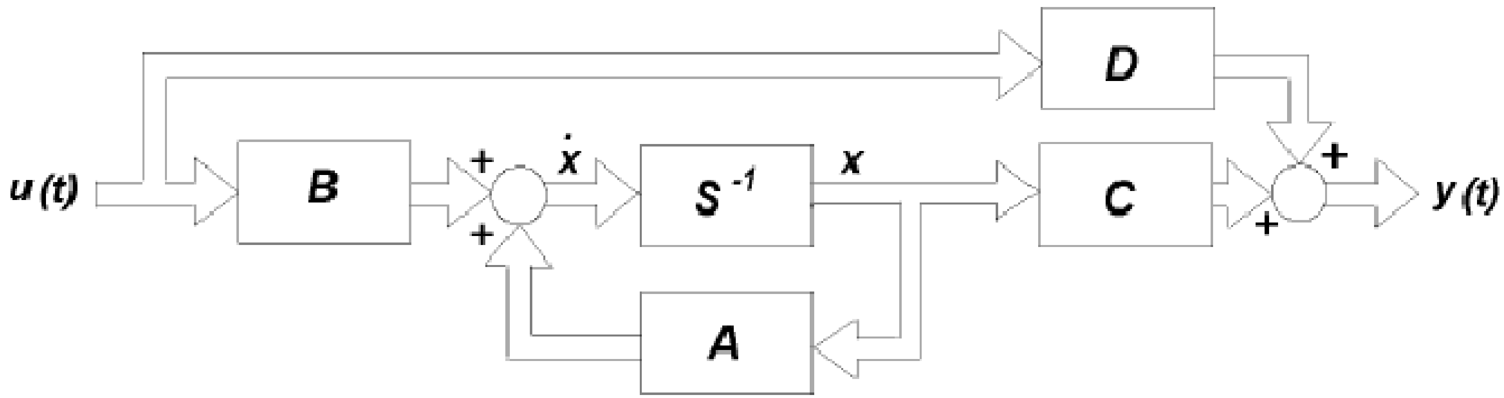
Block Diagram Representation of Linear Systems Described by State Equations

Step 1: Draw *n* integrator (S^{-1}) blocks, and assign a state variable to the output of each block.

Step 2: At the input to each block (which represents the derivative of its state variable) draw a summing element.

Step 3: Use the state equations to connect the state variables and inputs to the summing elements through scaling operator blocks.

Step 4: Expand the output equations and sum the state variables and inputs through a set of scaling operators to form the components of the output.



Example 1

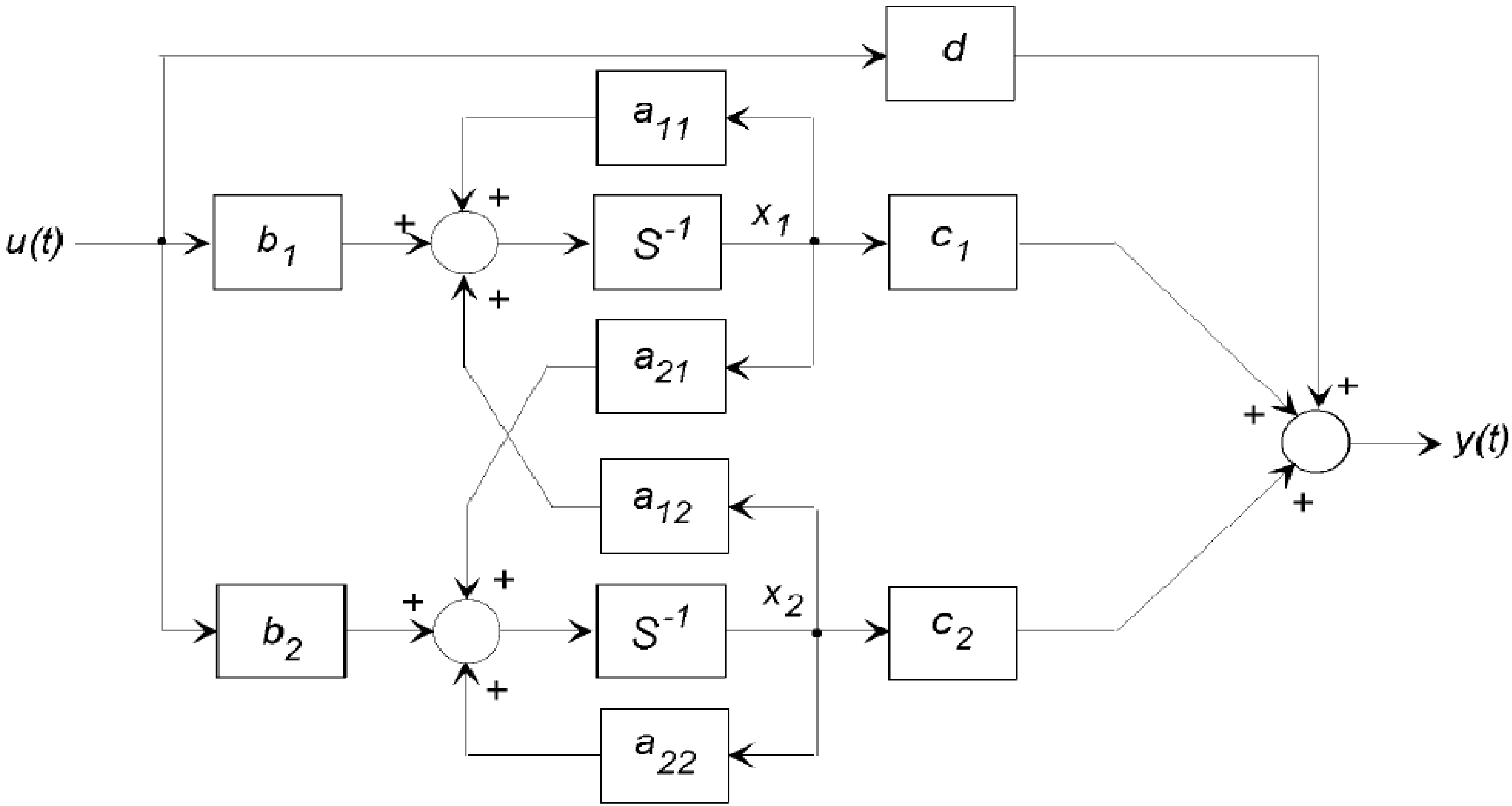
Draw a block diagram for the general second-order, single-input single-output system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + du(t).$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + du(t).$$

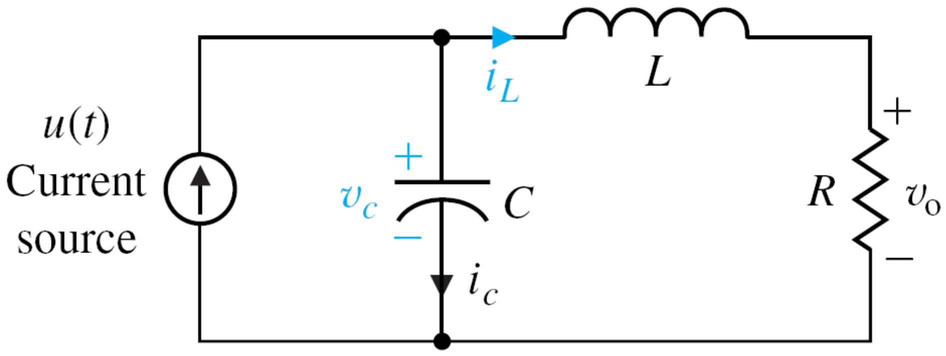


The overall modeling procedure developed in this chapter is based on the following steps:

1. Determination of the system order *n* and selection of a set of state variables from the linear graph system representation.

2. Generation of a set of state equations and the system **A** and **B** matrices using a well defined methodology. This step is also based on the linear graph system description.
3. Determination of a suitable set of output equations and derivation of the appropriate **C** and **D** matrices.

Consider the following RLC circuit



We can choose state variables to be $x_1 = v_c(t),\ x_2 = i_L(t),$

Alternatively, we may choose $\hat{x}_1 = v_c(t),\ \hat{x}_2 = v_L(t).$

This will yield two different sets of state space equations, but both of them have the identical input-output relationship, expressed by

$$\frac{V_0(s)}{U(s)} = \frac{R}{LCs^2 + RCs + 1}.$$

Can you derive this TF?

Linking state space representation and transfer function

- Given a transfer function, there exist infinitely many input-output equivalent state space models.
- We are interested in special formats of state space representation, known as *canonical forms*.
- It is useful to develop a graphical model that relates the state space representation to the corresponding transfer function. The graphical model can be constructed in the form of signal-flow graph or block diagram.

We recall Mason's gain formula when all feedback loops are touching and also touch all forward paths,

$$T = \frac{\sum_k P_k \Delta_k}{\Delta} = \frac{\sum_k P_k}{1 - \sum_{q=1}^{N} L_q} = \frac{\text{Sum of forward path gain}}{1 - \text{sum of feedback loop gain}}$$

Consider a 4th-order TF

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

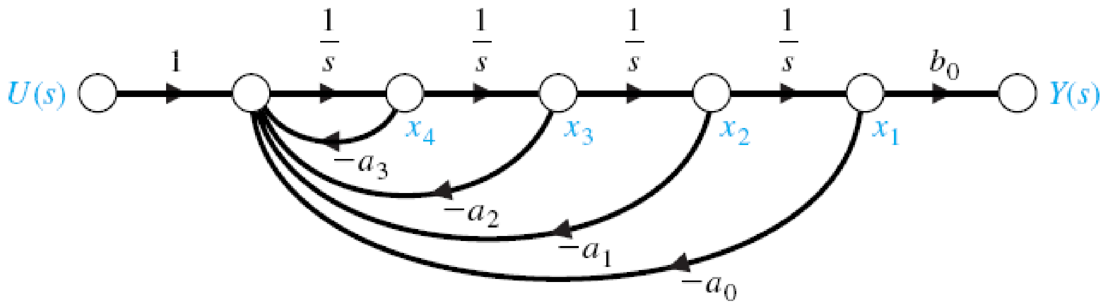
$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

We notice the similarity between this TF and Mason's gain formula above. To represent the system, we use 4 state variables Why?

Signal-flow graph model

This 4th-order system can be represented by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$



How do you verify this signal-flow graph by Mason's gain formula?

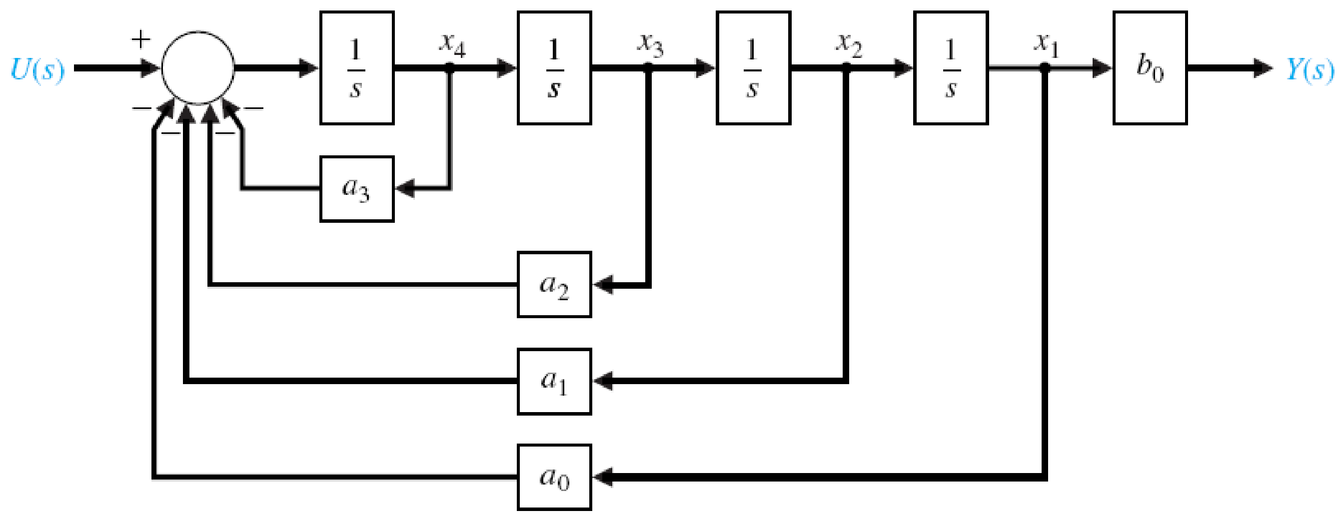
Block diagram model

Again, this 4th-order TF

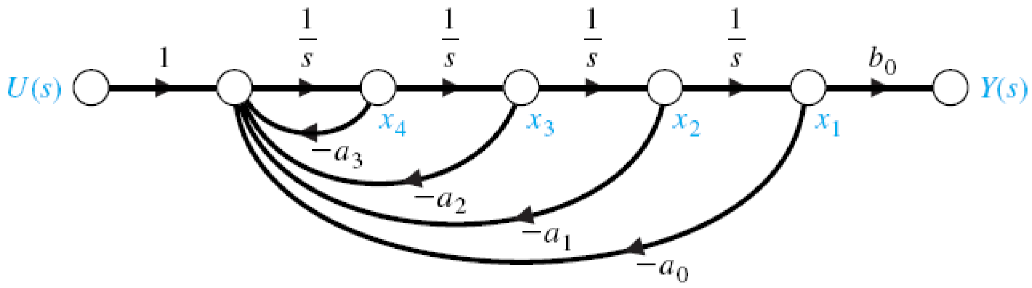
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

can be represented by the block diagram as shown



With either the signal-flow graph or block diagram of the previous 4th-order system,



we define state variables as $x_1 = {y}/{b_0}, \; x_2 = \dot{x}_1, \; x_3 = \dot{x}_2, \; x_4 = \dot{x}_3,$

then the state space representation is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u$$

$$y = b_0 x_1$$

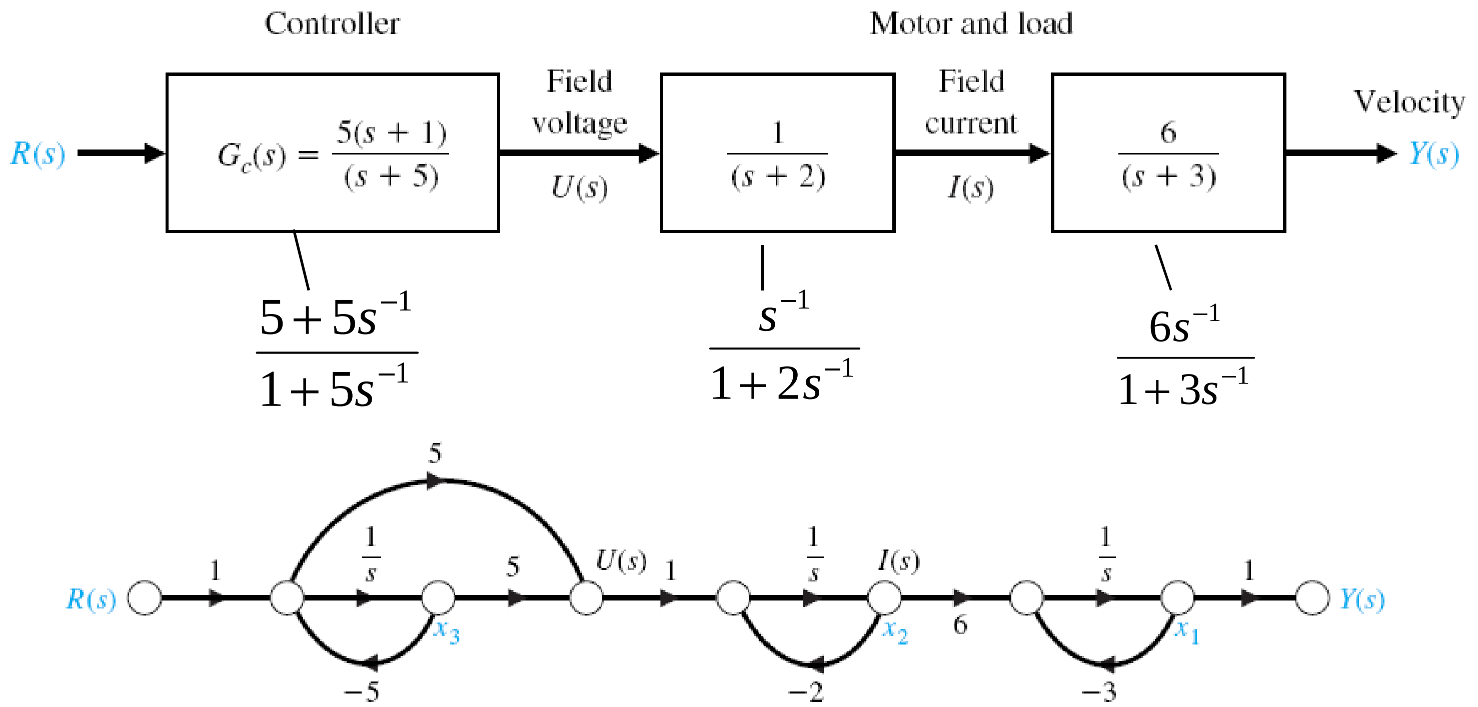
Writing in matrix form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

we have

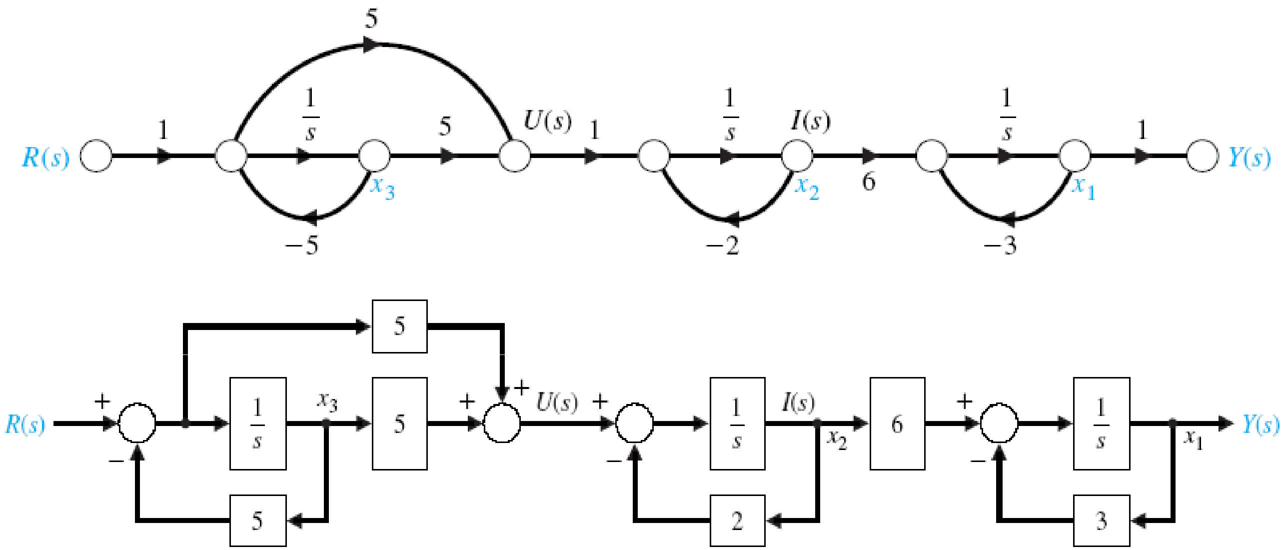
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} b_0 & 0 & 0 & 0 \end{bmatrix}, \qquad D = 0$$

When studying an actual control system block diagram, we wish to select the physical variables as state variables. For example, the block diagram of an open loop DC motor is



We draw the signal-flow diagraph of each block separately and then connect them. We select $x_1 = y(t)$, $x_2 = i(t)$ and $x_3 = (1/4)r(t) - (1/20)u(t)$ to form the state space representation.

Physical state variable model

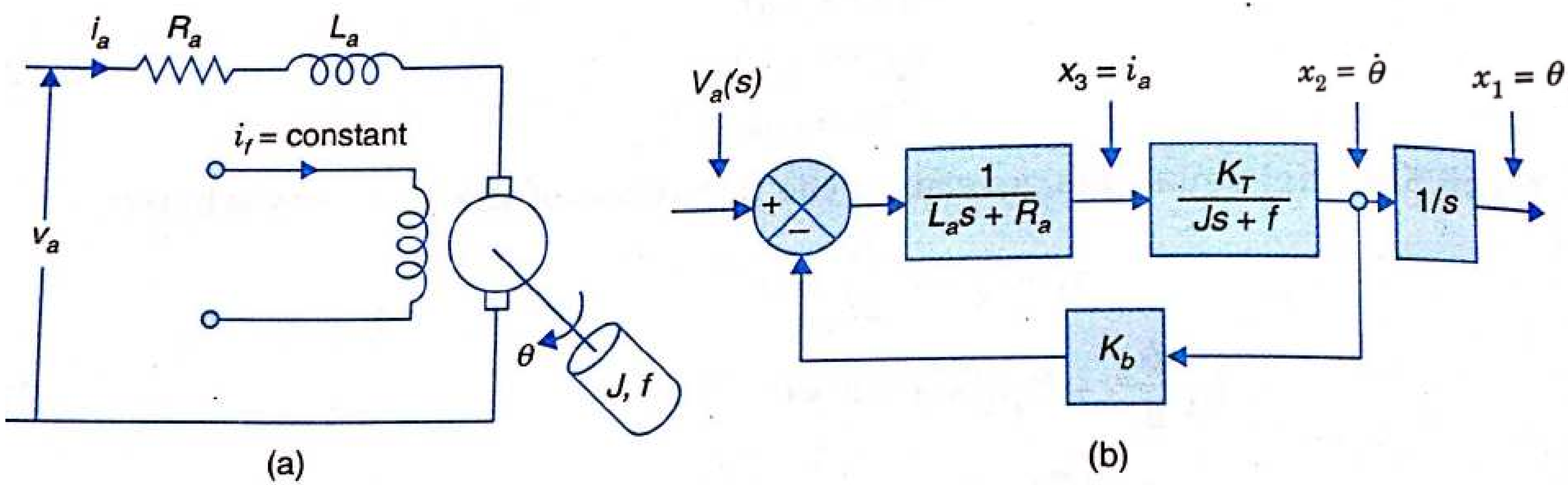


The corresponding state space equation is

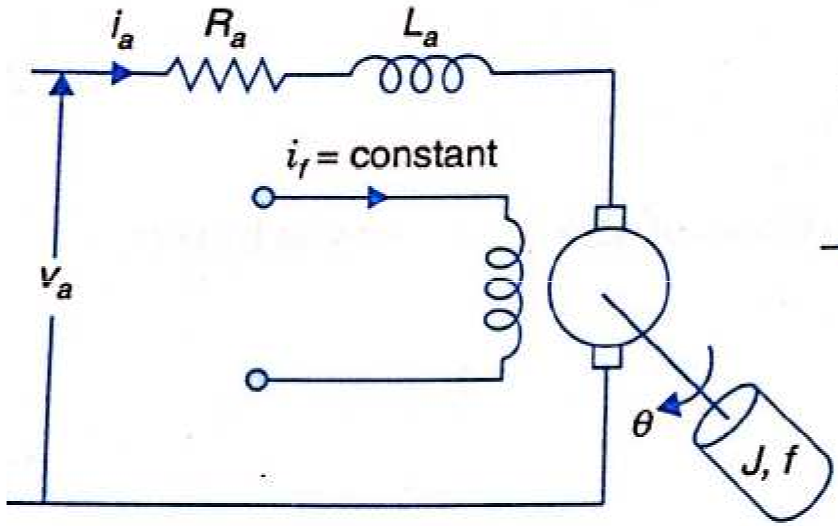
$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t)$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}$$

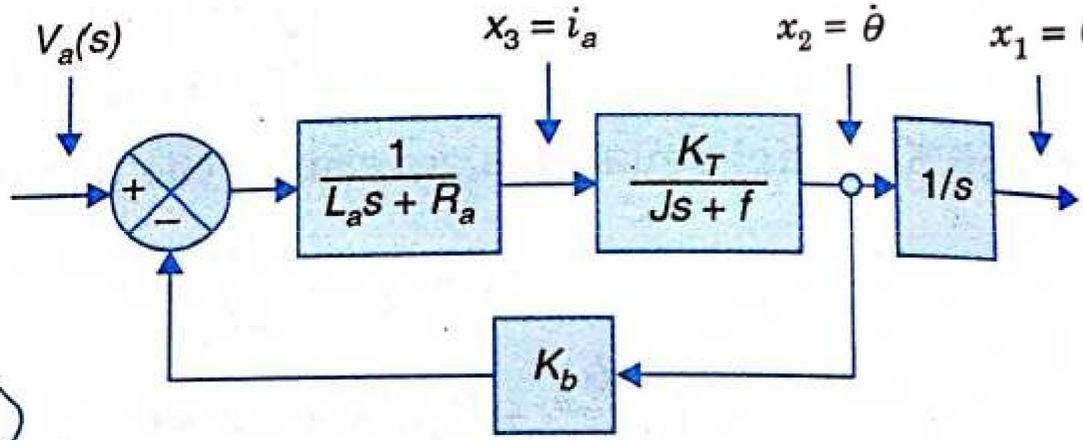
Electro Mechanical System



$$x_1 = \theta\,;\, x_2 = \dot{\theta}\ ;\text{ and } x_3 = i_a$$



(a)



(b)

$$\frac{1}{s}, \frac{K_T}{Js+f} \text{ and } \frac{1}{L_a s + R_a}$$

$$\dot{x}_1 = x_2$$

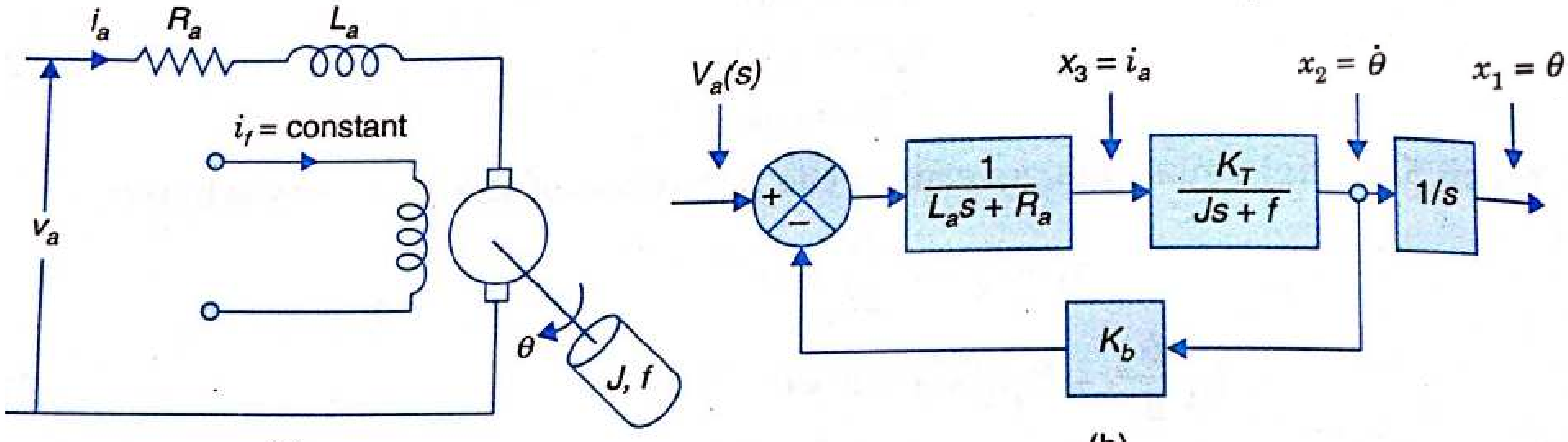
$$J\dot{x}_2 + fx_2 = K_T x_3$$

$$V_a - K_b x_2 = R_a x_3 + L_a \dot{x}_3$$

$$\dot{x}_1 = x_2$$

$$J\dot{x}_2 + fx_2 = K_T x_3$$

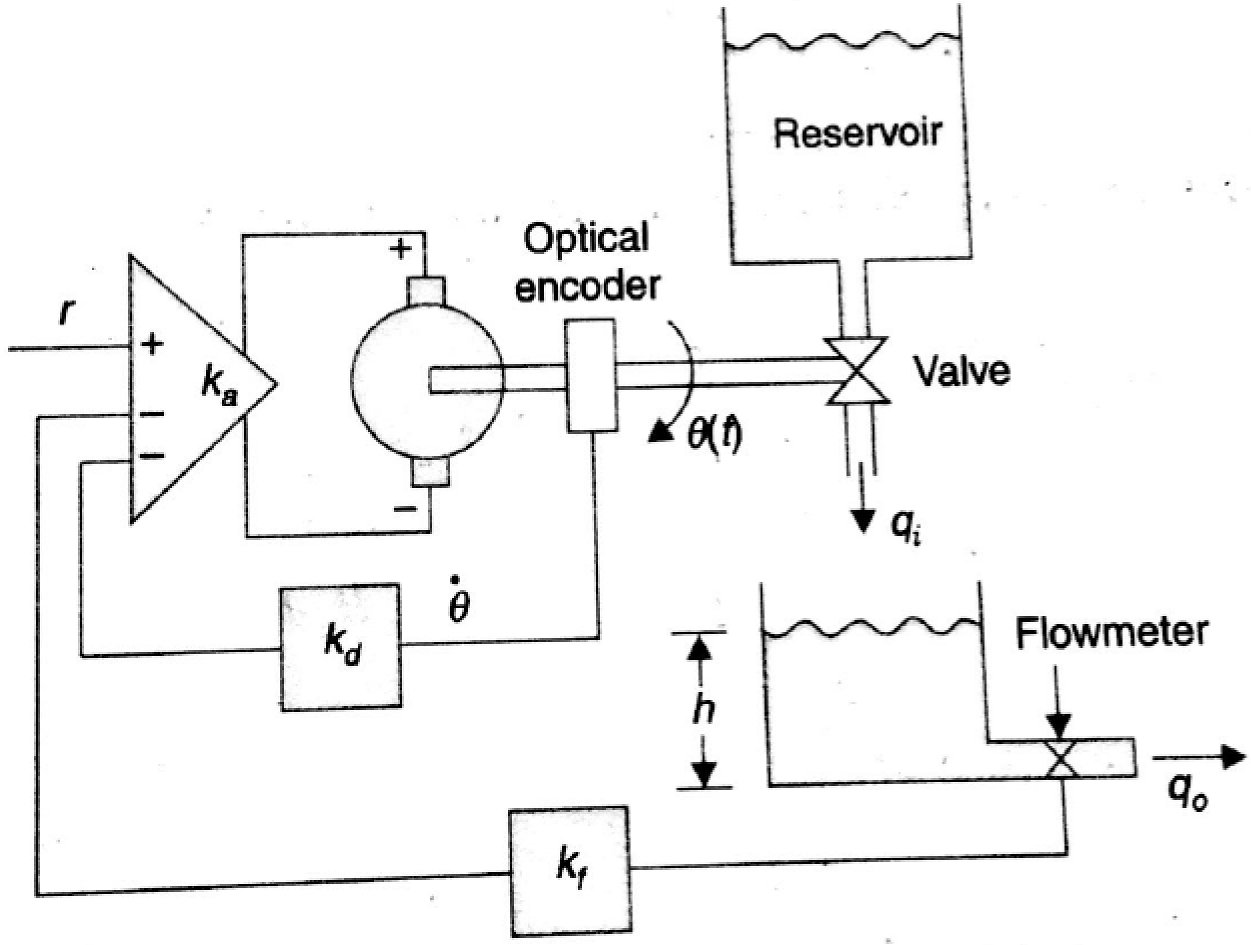
$$V_a - K_b x_2 = R_a x_3 + L_a \dot{x}_3$$



(a) (b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & K_T/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} v_a \qquad y = \theta = x_1 = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Control Flow

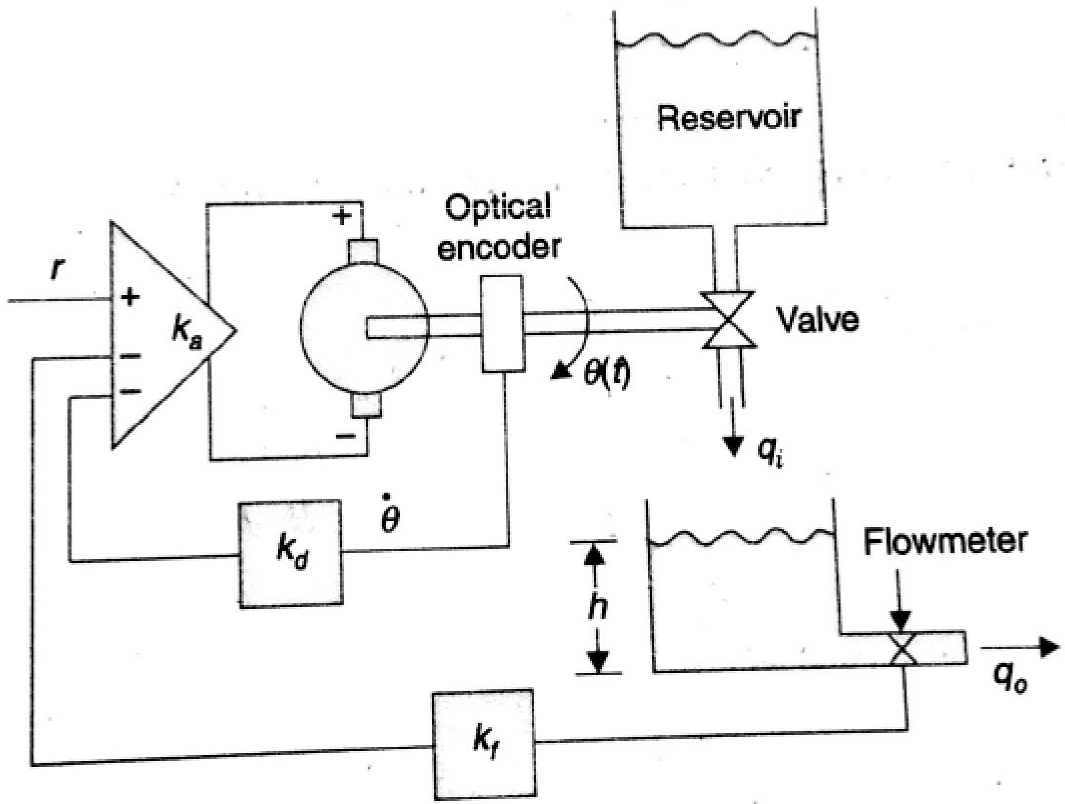


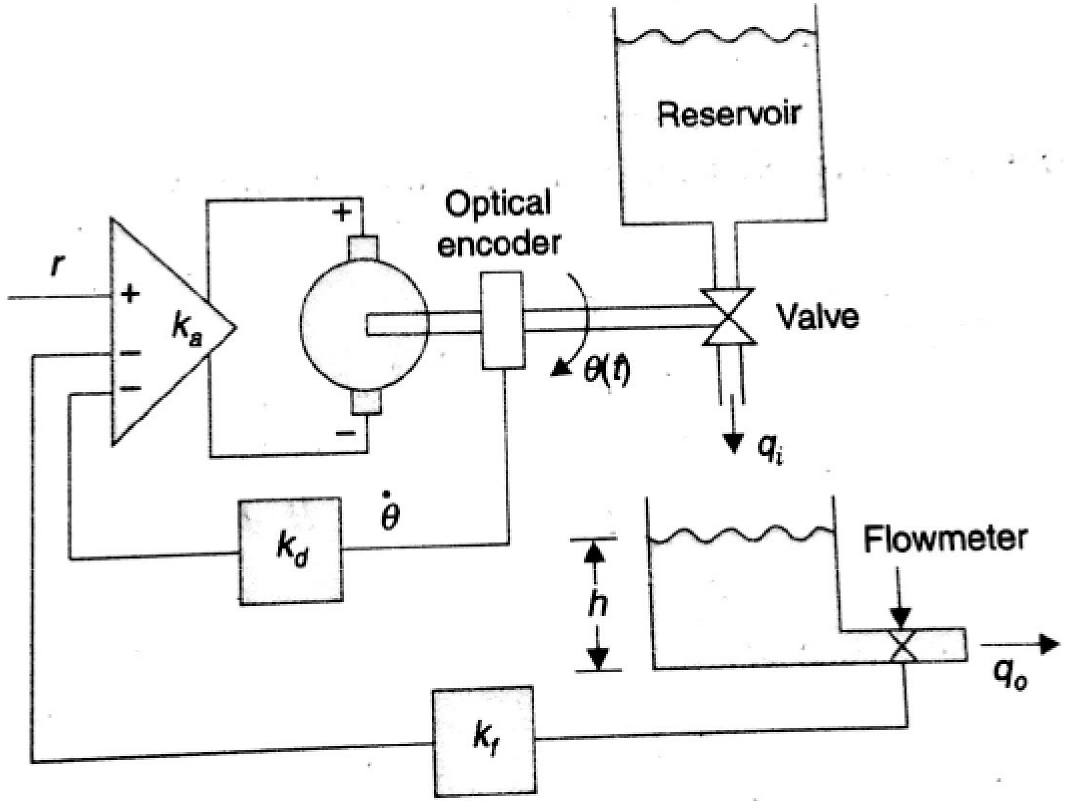
$k_a = 25,\ k_p = 1,\ k_d = 0.005$

$k_m = 5,\ J = 0.05,\ R_a = 1\ \Omega$

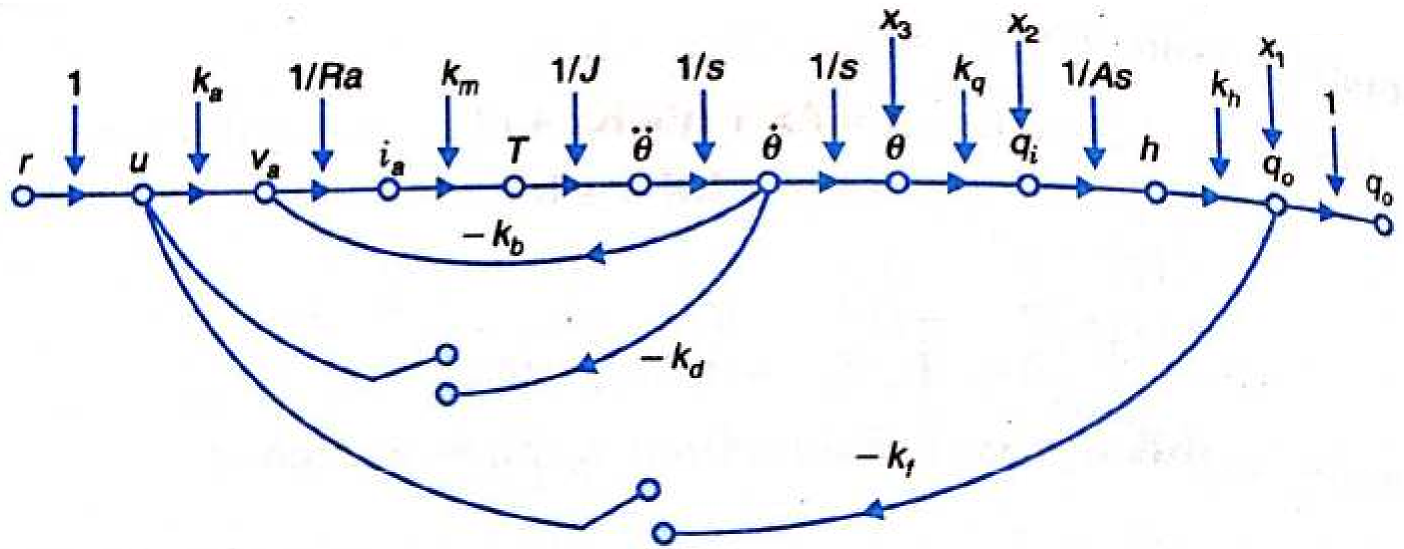
$q_i = k_q\theta,\ k_q = 8$, tank area $A = 50\ \text{m}^2$

$q_0 = k_h h,\ k_h = 225,\ k_f = 0.25.$





$$x_1 = q_0, x_2 = q_i, x_3 = \theta$$



$$\dot{x}_i = \frac{k_h}{A} x_2$$

$$\dot{x}_2 = k_q x_3$$

$$\dot{x}_3 = (rk_a - k_b x_3)\,\frac{k_m}{R_a J}$$

$$= \left(\frac{k_a k_m}{R_a J}\right) r - \left(\frac{k_b k_m}{R_a J}\right) x_3$$

$$y = q_0 = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_h}{A} & 0 \\ 0 & 0 & k_q \\ 0 & 0 & -\left(\frac{k_b k_m}{R_a J}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{vmatrix} 0 \\ 0 \\ \frac{k_a k_m}{R_a J} \end{vmatrix} [u]\,;\ u = r$$

$$y = [1\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State-Space Representations in Canonical Forms.

1- Controllable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1} \dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1} \dot{u} + b_n u$$

Special Case

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1} s + a_n}$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1}s + a_n}$$

$$y^{(n)} + a_1 y^{(n-1)} + \ldots\ldots + a_{n-1}\dot{y} + a_n y = bu$$

Assume

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\cdots\cdots\cdots$$
$$x_n = y^{(n-1)}$$

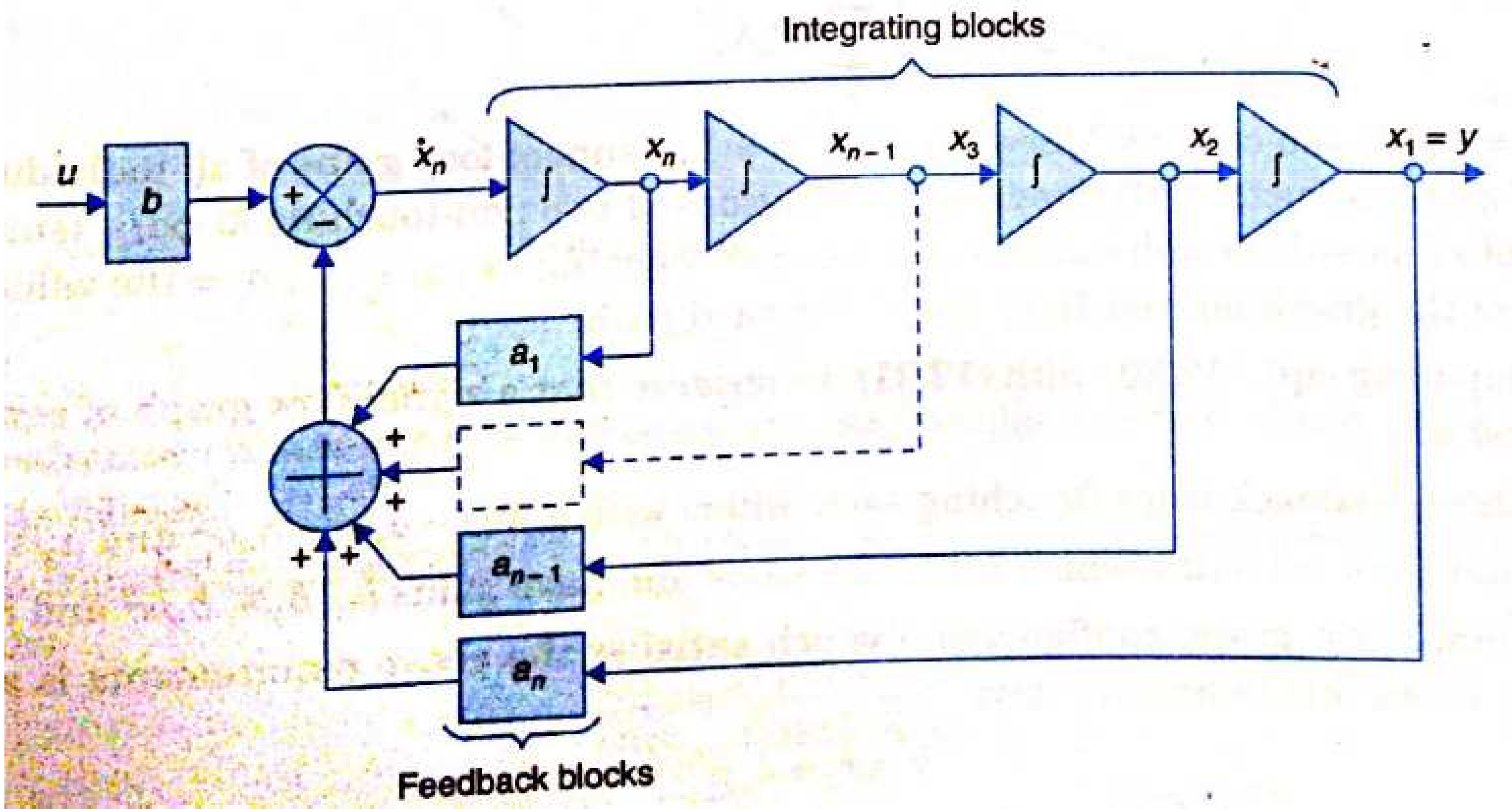
then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\cdots\cdots\cdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \ldots - a_1 x_n + bu$$

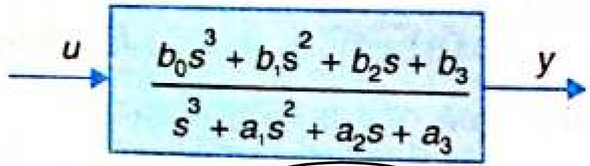
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\cdots\cdots\cdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \ldots - a_1 x_n + bu$$

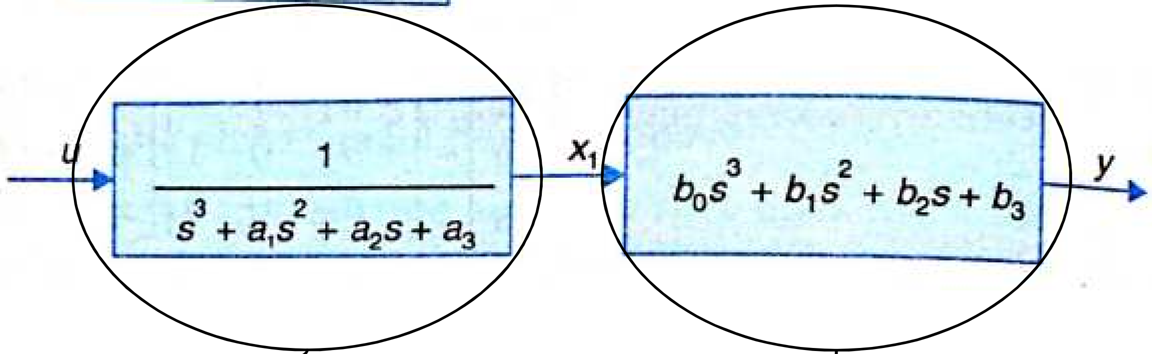


General Case



$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$$

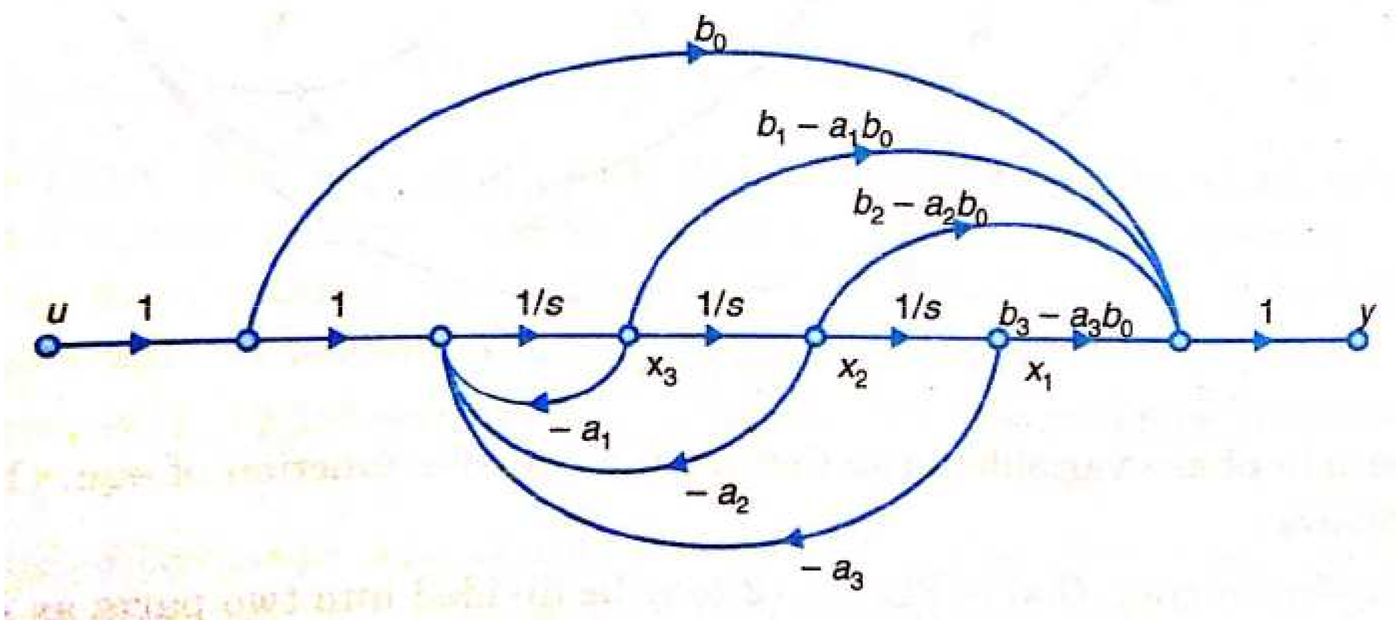
$$\frac{Y(s)}{X_1(s)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\begin{aligned} y &= b_0 \dddot{x}_1 + b_1 \ddot{x}_1 + b_1 \dot{x}_1 + b_3 x_1 \\ &= b_0(-a_3 x_1 - a_2 x_2 - a_1 x_3 + u) + b_1 x_3 + b_2 x_2 + b_3 x_1 \\ &= (b_3 - a_3 b_0) x_1 + (b_2 - a_2 b_0) x_2 + (b_1 - a_1 b_0) x_3 + b_0 u \end{aligned}$$

$$y = [(b_3 - a_3 b_0)\ (b_2 - a_2 b_0)\ (b_1 - a_1 b_0)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$


b0
b1 − a1b0
b2 − a2b0
u
1
1
1/s
1/s
1/s
b3 − a3b0
1
y
x3
x2
x1
− a1
− a2
− a3

Controllable Canonical Form General case

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \,\vdots\, b_{n-1} - a_{n-1} b_0 \,\vdots\, \cdots \,\vdots\, b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

2- Observable Canonical Form

Prove

$$\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+\cdots+b_{n-1}s+b_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

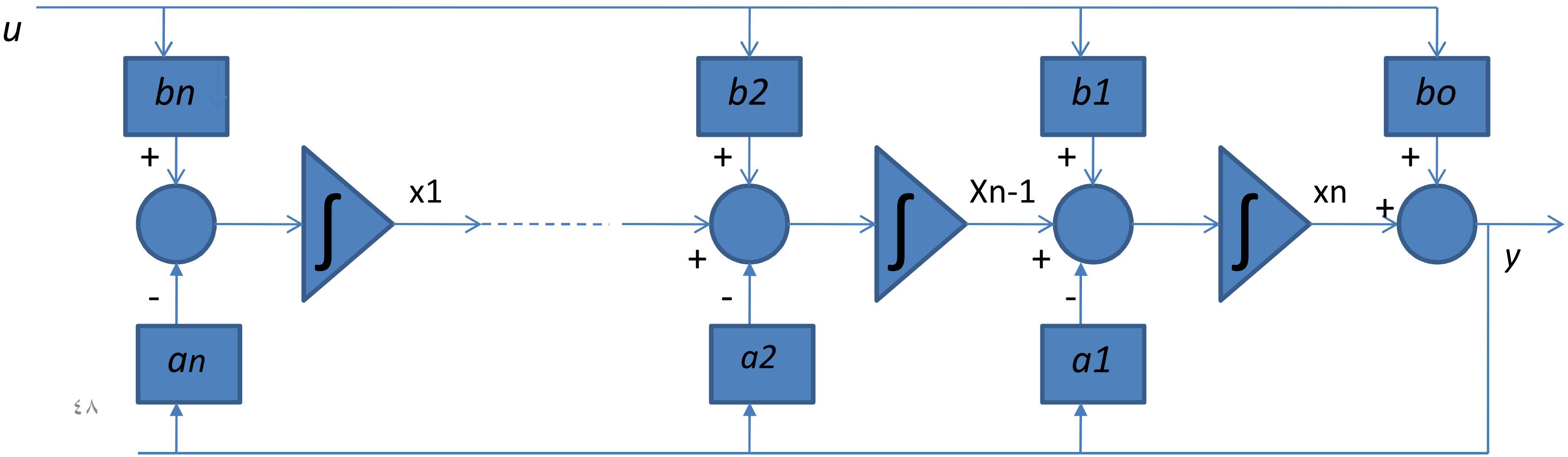
$$\frac{d^n}{dt^n}y+a_1\frac{d^{n-1}}{dt^{n-1}}y+a_2\frac{d^{n-2}}{dt^{n-2}}y+.....+a_ny=b_0\frac{d^n}{dt^n}u+b_1\frac{d^{n-1}}{dt^{n-1}}u+b_2\frac{d^{n-2}}{dt^{n-2}}u+.....+b_nu$$

rearrange

$$\frac{d^n}{dt^n}y=b_0\frac{d^n}{dt^n}u+\frac{d^{n-1}}{dt^{n-1}}(b_1u-a_1y)+\frac{d^{n-2}}{dt^{n-2}}(b_2u-a_2y)+.....+(b_nu-a_ny)$$

Integrate both side n times

$$y=b_0u+\int(b_1u-a_1y)dt+\iint(b_2u-a_2y)dt+.....+\underbrace{\iiint}_{n}(b_nu-a_ny)dt$$



General Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

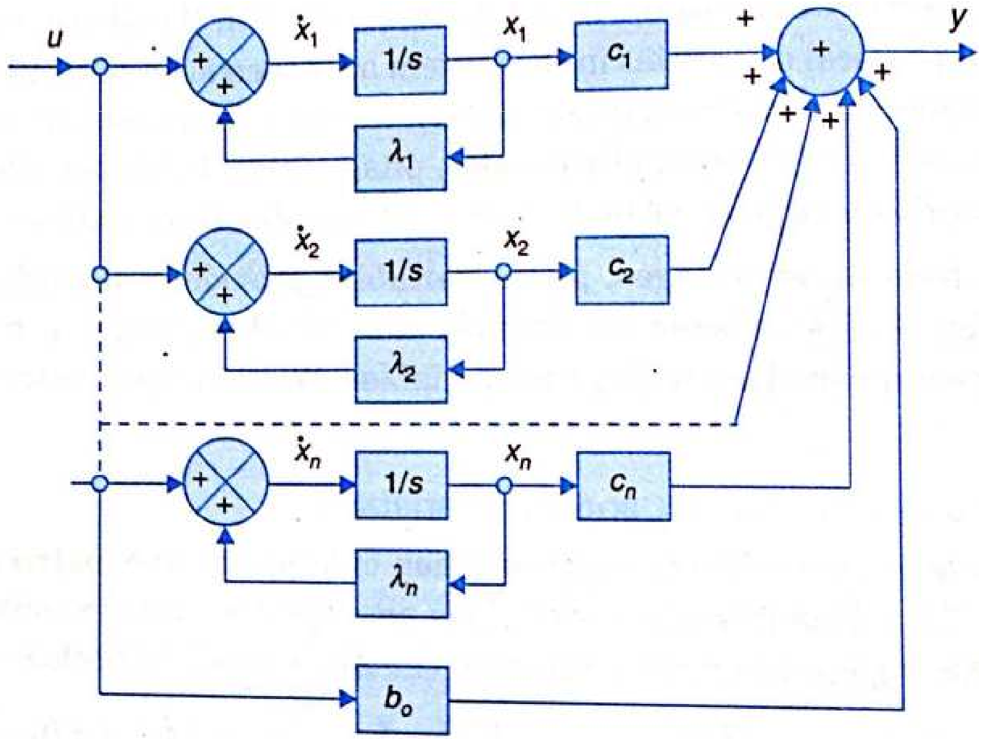
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

3- Diagonal Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

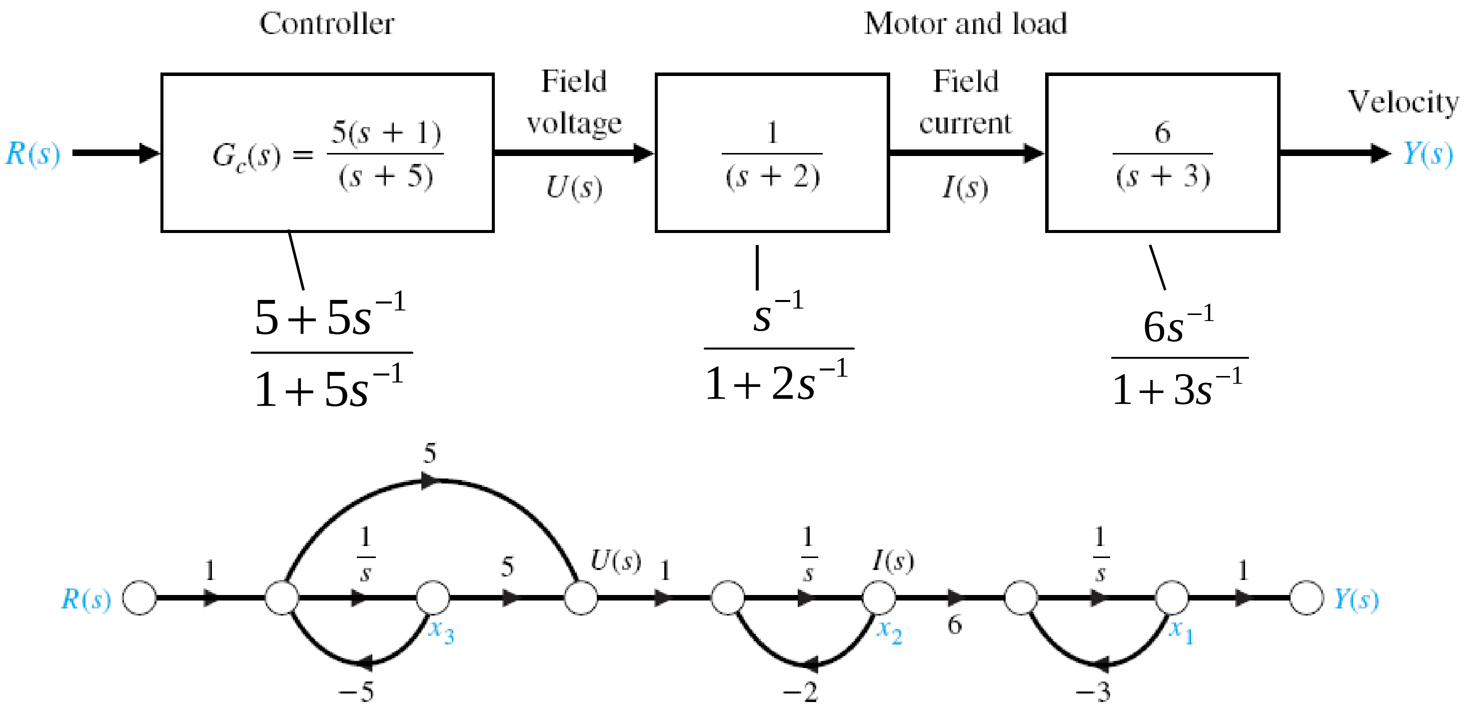
$$y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

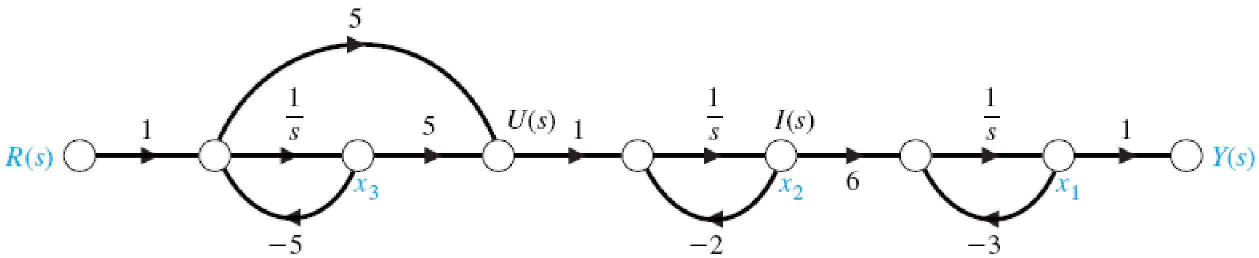
General Form

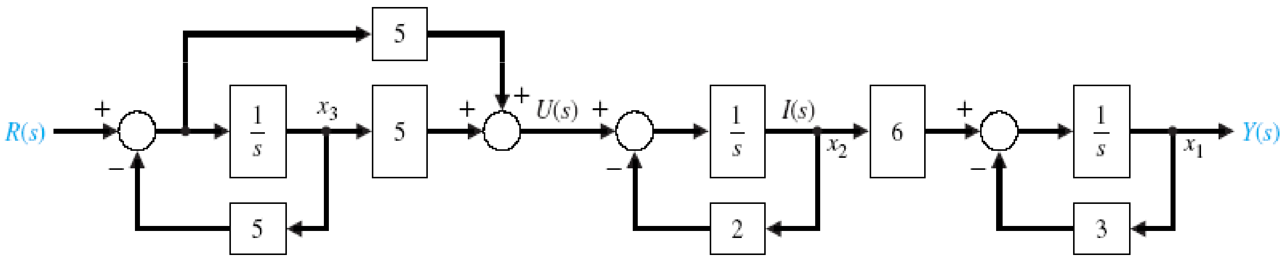
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & & 0 \\ & -p_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

4- cascade Form







The corresponding state space equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t)$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}$$

Examples

1- Consider the system given by

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Obtain state-space representations in the controllable canonical form, observable canonical form, and diagonal canonical form.

Controllable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Examples

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Observable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonal Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Examples

$$\frac{Y(s)}{U(s)} = \frac{10s + 10}{s^3 + 6s^2 + 5s + 10}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 10 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Examples

$$\frac{Y(s)}{U(s)} = \frac{25.04s + 5.008}{s^3 + 5.0325s^2 + 25.1026s + 5.008}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.008 & -25.1026 & -5.03247 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 . 24. & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Eigenvalues of an *n X n* Matrix A.

The eigenvalues are also called the characteristic roots. Consider, for example, the following matrix **A**:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The eigenvalues of **A** are the roots of the characteristic equation, or –1, –2, and –3.

Jordan canonical form

If a system has a multiple poles, the state space representation can be written in a block diagonal form, known as Jordan canonical form. For example,

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix}
=
\left[\begin{array}{ccc:cccc}
-p_1 & 1 & 0 & 0 & \cdots & & 0 \\
0 & -p_1 & 1 & \vdots & & & \vdots \\
0 & 0 & -p_1 & 0 & \cdots & & 0 \\
\hdashline
0 & \cdots & 0 & -p_4 & & & 0 \\
\cdot & & \cdot & & \cdot & & \\
\cdot & & \cdot & & & \cdot & \\
\cdot & & \cdot & & & & \cdot \\
0 & \cdots & 0 & 0 & & & -p_n
\end{array}\right]
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u
$$

Three poles are equal

$$
y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}
+ b_0 u
$$

State-Space and Transfer Function

The SS form

$$\begin{aligned} \dot{\mathrm{x}} &= \mathrm{Ax} + \mathrm{Bu} \\ \mathrm{y} &= \mathrm{Cx} + \mathrm{Du}. \end{aligned}$$

Can be transformed into transfer function

Tanking the Laplace transform and neglect initial condition then

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + B\mathbf{U}(s) \text{ and } \quad (1)$$

$$\mathbf{Y}(s) = C\ \mathbf{X}(s) + D\mathbf{U}(s) \qquad (2)$$

then

$$s\mathbf{X}(s) - A\mathbf{X}(s) = \mathbf{x}(0) + B\mathbf{U}(s)$$

$$s\mathbf{X}(s) - A\mathbf{X}(s) = \mathbf{x}(0) + B\mathbf{U}(s)$$

by neglecting intial condition then

$$(sI - A)\mathbf{X}(s) = B\mathbf{U}(s)$$

$$\mathbf{X}(s) = \left(sI - A\right)^{-1} B\mathbf{U}(s)$$

sub in 2

$$\mathrm{Y(s)} = \mathrm{C}\left(sI - A\right)^{-1} B\mathbf{U}(s) + D\mathbf{U}(s)$$

$$\mathbf{Y}(s) / \mathbf{U}(s) = G(s) = \mathrm{C}\left(sI - A\right)^{-1} B + D$$

State-Transition Matrix

We can write the solution of the ***homogeneous*** state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$ Laplace transform $$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) \Rightarrow \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

The inverse Laplace transform $$\mathbf{x}(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}(0)$$

Note that $$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3} + \cdots$$

$$\mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots = e^{\mathbf{A}t}$$

Hence, the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1}$

$$\mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots = e^{\mathbf{A}t}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

State-Transition Matrix $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$

where $\mathbf{\Phi}(t)$ is an $n \times n$ matrix and is the unique solution of

$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0) = \mathbf{I}$$

Where

$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$ Note that $\mathbf{\Phi}^{-1}(t) = e^{-\mathbf{A}t} = \mathbf{\Phi}(-t)$

If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix A are distinc $\Phi(t)$

will contain the n exponentials

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$$

$$\Phi(t) = e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & & & 0 \\ & e^{\lambda_2 t} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & e^{\lambda_n t} \end{bmatrix}$$

Properties of State-Transition Matrices.

1. $\Phi(0) = e^{\mathbf{A}0} = \mathbf{I}$
2. $\Phi(t) = e^{\mathbf{A}t} = \left(e^{-\mathbf{A}t}\right)^{-1} = \left[\Phi(\quad t)\right]^{-1}$ or $\Phi^{-1}(t) = \Phi(\quad t)$
3. $\Phi(t_1 + t_2) = e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$
4. $\left[\Phi(t)\right]^n = \Phi(nt)$
5. $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0) = \Phi(t_1 - t_0)\Phi(t_2 - t_1)$

Obtain the state-transition matri$\mathbf{\Phi}(t)$ of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\mathbf{sI} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\Phi(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\Phi^{-1}(t) = e^{-\mathbf{A}t} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

the ***NON- homogeneous*** state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \Longrightarrow \quad \dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

and premultiplying both sides of this equation by $e^{-\mathbf{A}t}$

$$e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)] = \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Integrating the preceding equation between 0 and t gives

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

or

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

unit-step function

$$u(t) = 1(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\,d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero, or $\mathbf{x}(0) = \mathbf{0}$, then $\mathbf{x}(t)$ can be simplified to

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] u$$

Prove Transfer function of the given ss

$$\frac{Y(s)}{U(s)} = \frac{10s + 10}{s^3 + 6s^2 + 5s + 10}$$

Solution

$$G(s) = C(sI - A)^{-1} B + D$$

Relation of Different SS Representations of the Same System

For a given system G(s) has two different ss representations

$$\text{Rep.1}: \mathrm{M}_1 : \mathbf{x}(t) = A_1\mathbf{x}(t) + B_1\mathbf{u}(t)$$
$$\mathbf{y}(t) = C_1\mathbf{x}(t) + D_1\mathbf{u}(t)$$
$$\text{Rep.2}: \mathrm{M}_2 : \mathbf{x}(t) = A_2\mathbf{z}(t) + B_2\mathbf{u}(t)$$
$$\mathbf{y}(t) = C_2\mathbf{z}(t) + D_2\mathbf{u}(t)$$

Let Z=T x

Where T is the transformation matrix between x and z

For example

$$\textit{take}$$
$$x_1 = y\,, \qquad \mathrm{z}_1 = y$$
$$x_2 = \dot{y}, \qquad \mathrm{z}_2 = \dot{y} + y$$
$$x_3 = \ddot{y}, \qquad \mathrm{z}_1 = \ddot{y} + \dot{y}$$

take

$$x_1 = y\,, \qquad z_1 = y$$

$$x_2 = \dot{y}, \qquad z_2 = \dot{y} + y$$

$$x_3 = \ddot{y}, \qquad z_3 = \ddot{y} + \dot{y}$$

then

$$z_1 = x_1$$

$$z_2 = x_2 + x_1$$

$$z_3 = x_3 + x_2$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{z} = T\,\mathbf{x}$$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Sub. By z=Tx in rep. 2

$$\dot{z}(t) = T\mathbf{x}(t) = A_2 T\mathbf{x}(t) + B_2\mathbf{u}(t) \text{ mutiply by } \mathrm{T}^{-1}$$

$$T^{-1}\dot{z}(t) = T^{-1}T\dot{\mathbf{x}}(t) = T^{-1}A_2 T\mathbf{x}(t) + T^{-1}B_2\mathbf{u}(t)$$

$$y(t) = C_2 T\mathbf{x}(t) + D_2\mathbf{u}(t)$$

Compare with M1;rep.1

$$\text{Rep.1}: \mathrm{M}_1 : \mathbf{x}(t) = A_1\mathbf{x}(t) + B_1\mathbf{u}(t)$$
$$\mathbf{y}(t) = C_1\mathbf{x}(t) + D_1\mathbf{u}(t)$$

then

$$A_1 = T^{-1}A_2 T \qquad A_2 = TA_2 T^{-1}$$
$$B_1 = T^{-1}B_2 \qquad B_2 = TB_1$$
$$C_1 = C_2 T \qquad C_2 = C_1 T^{-1}$$
$$D_1 = D_2 \qquad D_1 = D_2$$

State-Space Diagonalization Function

Eign values and eign vectors

Definition: for a given matrix A, if ther exist a real (complex) λ and a corresponding vector v≠0, such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

Then λ is called eign value and v is the eign vector

i.e.

$$(A - \lambda I)\mathbf{v} = 0$$

And since v≠0

Then

$$(A - \lambda I) = 0$$

i.e

$$\det(A - \lambda I) = 0$$

Eigenvalues of an *n X n* Matrix A.

The eigenvalues are also called the characteristic roots. Consider, for example, the following matrix **A**:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The eigenvalues of **A** are the roots of the characteristic equation, or –1, –2, and –3.

Example

$$A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

$$\text{then the eign value is the solution of } |\lambda \mathrm{I} - \mathrm{A}| = 0$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -8 & (\lambda + 2) \end{vmatrix} = 0$$

$$\lambda^2 + 2\lambda - 8 = 0 = (\lambda + 4)(\lambda - 2)$$

$$then$$

$$\lambda_1 = -4 \quad and \quad \lambda_2 = 2$$

Eign vectors are obtained as

$$at\ \lambda = \text{-}4$$

$$(\lambda_1 \mathrm{I} \text{ - } \mathrm{A})v_1 = 0$$

$$i.e.$$

$$\begin{bmatrix} -4 & -1 \\ -8 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\therefore \quad v_{12} = -4v_{11}$$

$$let \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$at\ \lambda_2 = 2$$

$$(\lambda_2 \mathrm{I} \text{ - } \mathrm{A})v_2 = 0$$

$$i.e.$$

$$\begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$\therefore \quad v_{22} = 2v_2$$

$$let \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eign vector matrix

$$V = [\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_n]$$

For all eign values and vectors

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i; \quad i = 0,1,\ldots,n$$

These equations can be written in matrix form

$$A\mathbf{V} = \mathbf{V}\Lambda$$

where

$$\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2 \cdots \mathbf{v}_n]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = diag\{\lambda_i, i = 1,2,\cdots,n\}$$

thus

$$A = \mathbf{V}\Lambda\mathbf{V}^{-1}$$

$$\Lambda = \mathbf{V}^{-1}A\mathbf{V}$$

thus

$$e^{At} = \phi(t) = I + At + A^2 \frac{t^2}{2!} + \dots$$

$$e^{At} = Ve^{\Lambda t}V^{-1}$$

$$e^{\Lambda t} = \phi(t) = I + \Lambda t + \Lambda^2 \frac{t^2}{2!} + \dots$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} = diag(e^{-\lambda_i t}, i = 1,2,\dots,n)$$

Then for a given system has a system matrix A and a state vector **X**

The diagonal system matrix Ad and state **Xd**

$$A_d = \Lambda = T^{-1}AT$$

$$\mathbf{x} = T\mathbf{x}_d \; ; \; \mathbf{x}_d = T^{-1}\mathbf{x}$$

$$T = V = eign \quad vector \quad matrix$$

$$A_d = V^{-1}AV$$

$$B_d = V^{-1}B_1$$

$$C_d = C_1 T$$

$$D_1 = D_2$$

Example 2: find the transformation into diagonal form and the state transition matrix of example1

$$\Lambda = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$e^{At} = V e^{\Lambda t} V^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}^{-1}$$

$$e^{At} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}$$

Discus how to obtain the transformation matrix between two representation

Diagonal Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Alternative Form of the Condition for Complete State Controllability.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times r$ matrix

If the eigenvectors of **A** are distinct, then it is possible to find a transformation matrix **P** such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \lambda_n \end{bmatrix}$$